# 1. INTRODUCTION TO THE AUTOMATIC CONTROL

## 1.1. The essence and object of the automatic control

It could be imagined that the man has thought of simplification of its work

from the beginning of the ages - how to get the water into its house in a more simple way, how it were possible to move more goods at once, how to make other works more quickly, could it be possible to make these jobs so that they could be done without assistance. The objective of it was the transformation from the work to the mechanised work and so has the technology developed during centuries – first the methods were found how to lease the force /energy) requiring works to the machines and thereafter the possibilities to leave the information processing to the machines as well. Although the transformation of the energy level to the machines has happened already long time ago, the processing of the information by machines

has taken place on a quite primitive level until the development on the computer technology. The development of the computer technology has speeded up the development of many other branches of sciences, the development control technology included which in turn enabled to entrust to the machines activities requiring more decisiveness.

For the description of such systems, where the right to decide is trusted to the machines the **term automated system** was applied. This term is inherited from the Greek, where the word *automatos* means self-acting in English. By different literature sources, the systems are divided differently by the level of automation, by the grade of automation [1], for instance, that which is described in the able 1.1

Table 1.1. Classification of system by the grade of automation

| *Grade* | *Replaced function* | *Example* |
|---|---|---|
| A (0) | Missing | Hand working tools, spinning reel as example |
| A (1) | Energy | Drill |
| A (2) | Skilfulness | Lathe with feeding |
| A (3) | Diligence | Automatic lathe with open control loop and repeated production cycle |
| A (4) | Decision | Lathe with closed control cycle and computer control |

Such a division is provisional; therefore, it is suitable sometimes to use simpler on the control-based division. The **control** is an information processing process, expressed purposeful rearrangement of some action. As an action technical, as well biological and social processes could serve. [2]. Focusing on the technical processes one my distinguish in them

**hand control**, corresponding to the automation level(0);
**automated control**, matching A (1)  to A(3);

and **automatic control**, matching grades A(3) and higher.

As technical systems, a set of equipment functioning together for fulfilment of the objective of the work is meant .The originality of the work is described by the physical properties of the equipment, which could be described by mathematical equations, after solution of which is it possible to analyse the system. If the system does not comply with required conditions then is it necessary to operate the system in such a way that it will comply with the requirements. In case if the system is able to apply such impacts without man involvement, one has to deal with **automatic control system**

## 1.2. Recitals

The automatic control systems are divided in to two groups – open loop systems and closed loop systems. In the **open loop systems,** preliminary defined mathematical models control the process, without controlling the satisfaction of the results of controlled process to the required ones. Such a control suits simple systems, so as the undesirable influences or **disturbances** are affecting the results of the process. The **process noise** arises from the physical patterns influencing the process, affecting its behaviour. The **measurement noise** arises from the measurement errors of the measuring devices or sensors. In general, the first of them is a random value, which is not possible to forecast, however the measurement error could be taken into account preliminary and influence the system in such a way that in the **control action** (influence of the control to the system) the arising error will be compensated. Such control is called the **compensation of disturbances.**

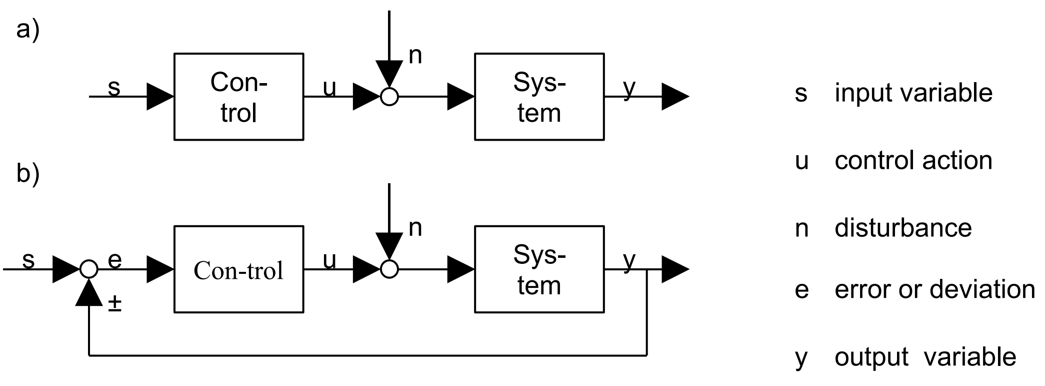



Figure 1.1. Automatic control systems a) open loop system, b) closed loop system

In the closed loop systems, the process is controlled inspecting the compliance of the results with the given criteria or the feedback about the results is functioning. The process is controlled by the error or in dependence on the difference of the actual and desired result. The controlled process is called **control object,** in general, and its result - **output** *y***.** The device, which is forming the control action, is called **control device** or also **regulator.** The system is influenced by input variables, which are inherited form outside of the system. These inputs are **set point** *s,* which determines what is required from the system and the disturbances *n* which are disturbing the process functioning

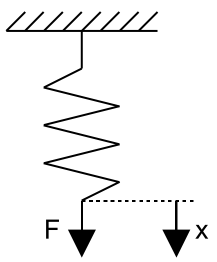
In the Figure 1.1. an automatic control system consisting from blocks and directed influences between blocks is shown. The blocks are called **transfer links** and the influences – **signals.** If to examine one individual block, then the following relation holds:

$$y = W \cdot s \quad . \tag{1.1}$$

The quantity W is called **transfer function** and it describes the dependance of output variable on the input variable..

Figure 1.2. Transfer link

*Example 1.1.*




If to apply a force to the spring, then as a result of it the length of the spring is changing. Expressing it based on the equation (1.1)

$$x = W \cdot F \quad .$$

From the Hook’s law the following expression is known:

$$F = k \cdot x$$

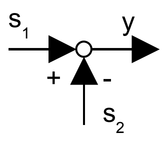
and from these expressions it is possible to conclude, that

$$W = \frac{1}{k}$$

Figure 1.3. A spring .

In the block diagrams or flow charts three types of signal processing elements are present (addition link, branching link and multiplication link) and three types of main or simple structures (series structure, parallel structure and feedback loop) [3].

.




For the adding link the following mathematical relation holds

$$y = s_1 - s_2 \ . \tag{1.2}$$

Figure 1.4. Addition link

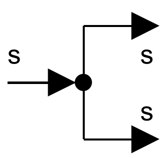
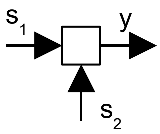


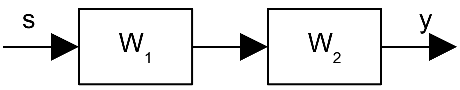
Figure 1.5.
Branching link




Mathematical presentation of the multiplication link is the following

$$y=s_1 \cdot s_2 \ . \qquad (1.3)$$

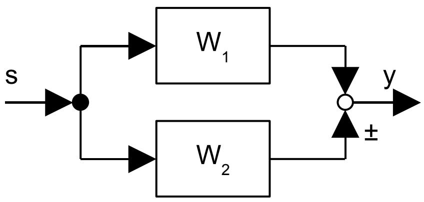
Figure .1.6.
Multiplication link




For the series structure the followings holds

$$y=W_1 \cdot W_2 \cdot s \ . \qquad (1.4)$$

Figure 1.7. Series structure




The output of the parallel structure is expressed as

$$y=(W_1 \pm W_2) \cdot s \qquad (1.5)$$

Figure 1.8. Parallel structure

Feedback structure is presented mathematically

$$y=\frac{W_1}{1 \mp W_1 \cdot W_2} \cdot s \qquad (1.6)$$

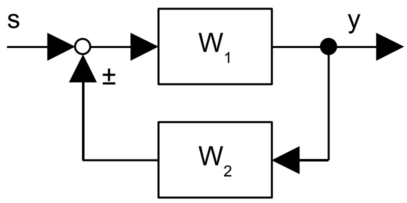



Figure 1.9. Feedback structure

The transfer functions are considered more detailed in the next chapter.

The transfer functions are used for the description of the dependence of one output variable on the one input variable. In case of more complicated systems, where it is needed to follow multiple outputs depending on multiple inputs simultaneously or

**multilinked systems (?),** the calculation using transfer functions becomes complicated. The calculation becomes easier if to describe the system by means of the **state equations** (look also division 2.3).

$$\dot{x}=A\cdot x+B\cdot u$$
$$y=C\cdot x+D\cdot u \quad , \qquad (1.7)$$

where

**A** – state matrix of the system;
**B** – input matrix;
**C** – putpu matrix;
**D** – disturbance’s matrix;
**x** – state vector;
**u** – input vector;
**y** – output vector.

As a **state** of the system the aggregated whole of all **state variables** of the system is called. For instance, while describing the state of a spring we have there two variable quantities – the force and the shift. By the force, we have to deal with **input variable,** by the shift – with **output variable**. At the same time there could be more state variables which are presented precisely or approximately, yet the question about their suitability to be considered as output variables is determined by the following criteria: as output variable such state variable could be, which could be precisely measured and dos not belong into input variables simultaneously. These criteria could be applied also to the transfer functions:

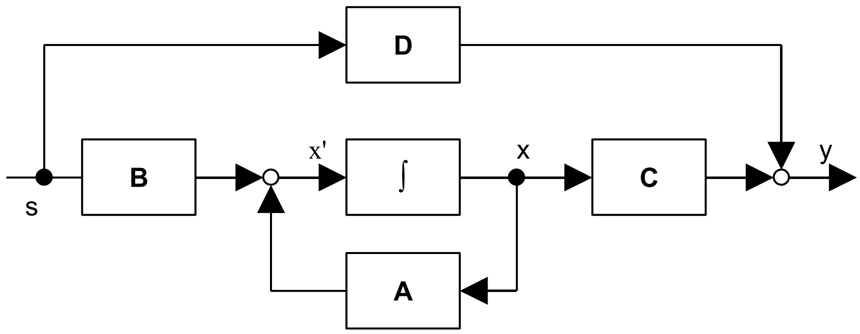



Figure 1.10. Graphical presentation of state equations

## 1.3. The components of an automatic control system

Although the automatic control systems could be divided into control objects and control devices in general, it is not possible to determine, which devices are used in the system. Also it necessary to know the so-called background components. to understand the influence of possible disturbance.

Therefore, beginning the description from the control object or device, or from the process, which is to be controlled, one of the components will be the controllable device, electric motor, for example. This device is influenced by control actions and also by distrubances, as a result of which the output of the control object or the result of the process changes. To control the result it has to be measured, therefore also the measuring device or sensor, to which, dependent on the structural individualities, systematic or random errors are influencing, belongs to the control object.  A random error could be influence of the temperature, presence of outer fields etc. System errors are contingent upon the accuracy of the measuring device.

If the signal of the measuring device are not suitable for the control device, digital computer cannot process analogous signals, for instance, the it will be required to add to the measurement device a signal converter, which changes the physical properties of the signal and/or makes the quantification of the signal. In this energy conversion process the errors similar to the measurement devices appear.

As a centre of the automatic control system the control device could be considered, which is replacing the man, deciding about the output of the system, in case if one has to deal with feedback system, or, in case of open system, takes into consider all possible deviations. In addition, to this part of the system, different disturbances are influencing and similarly to the adjustment of the sensor signal to the control device could be needed to adjust the control signal to the control object. Which means to have another signal converter in the system.

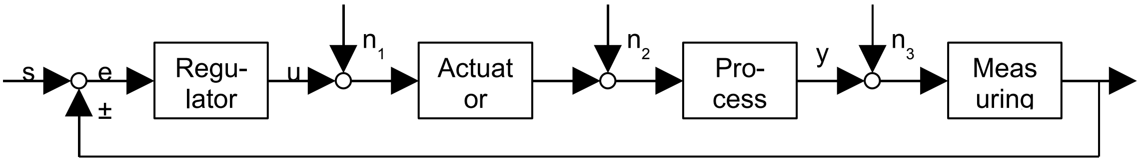


Figure 1.11. Automatic control system

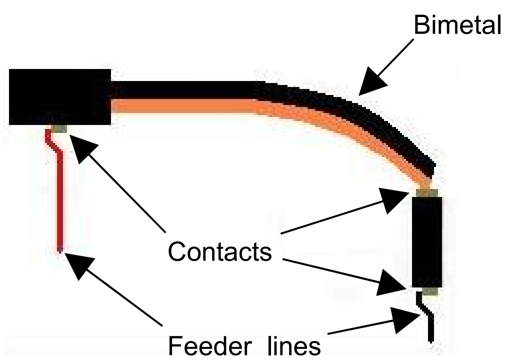
Although, it could not be left out of attention, that the set value, with which the order for the control system is submitted has certain error. However, the most complications are introduced to the system by the time. Namely, for the processing of the signal in each link takes certain time as well as the motion of the signal in the connection lines between links which is a serious problem in large factories. Therefore, the control device regulates mainly the system state that has already passed.

### 1.3.1. Regulator

Regulator is so-called crucial element of the automatic control system, which forms the control action based on deviation. Therefore, the adding or comparing link before the regulator belongs to the control device. Also belongs to this association the element for producing the set value, for which a simple switch could be serve in most simple cases (to switch the process on or off), or potentiometer by sliding contact of which the set value could be changed smoothly.

Real implementation of the regulator depends directly on the tasks and requirements for the regulation quality. This could be implemented by analogue computers, digital computers, and relays or by some other means. The regulator could be an individual device or could be incorporated into the actuator.

It is most simple to represent a temperature regulator of the relay action




Such regulator could be used in the electric radiators. Into the feeder line of a radiator, a bi-metal element will be placed, which opens and closes the contacts depending on the temperature. It is possible due to the different linear expansion coefficients of the metals used in the bi-metal element, which determines the binding of the element

Figure 1.12. Bimetal-regulator

From the incorporated regulators the controllers could be mentioned, where the regulators are implemented by software, or the microprocessor decides based on the information obtained from the measuring device about the suitable control action. The LOGO! Controller could be put into operation similarly to the bi-metal regulator. .

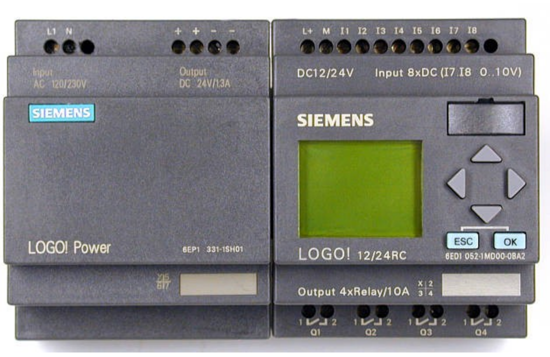



Figure 1.13. Siemens LOGO!Controller

### 1.3.2. Actuator


Actuator is that part of the control system, which amplifies the control action and converts it acceptable for the controlled device. Regulator, represented in the Figure 1.12. is an actuator at the same time, but it is only possible by small power control systems In case of large power control systems the converters must be used, so as the regulators are constructed as small power devices, to provide the minimal dimensions and reduction of the losses. For instance, if the output signal of the regulator lies in-between 0…5 V and the rated voltage of the controlled electric motor is 400 V, then for their match an actuator or a (semiconductor) converter has to be used, which has to compose the internal variables (opening angels of the thyristors) from the control action, on the bases of which the supply voltage of the motor will be formed.

The regulator could be incorporated into the converter, what mist the margin between control device and control object

Figure 1.14. ABB ACS-800 converter

### 1.3.3. Process

If generalised, a process is a device, which is subject to control in general. This is the only part of the automatic control system, the parameters of which could not be changed; therefore, it forms the base of the control system. and the system will be built around it.

Parameters of all other parts of the system could be changed or they would be replaced. For instance, let the process be regulation of the speed of an asynchronous machine, to which different regulators (continuous, discrete, etc.), actuators (frequency converter, rheostat etc), and measuring devices (velocity transducer, incremental encoder etc) could be chosen. By selection of these elements, one has to decide how they will match mutually and with the machine. The machine will be replaced if it becomes clear that it does not match the given task and in this case, all elements will be reselected.

### 1.3.4. Measuring device.

The measuring device is used for obtaining information about the system parameters. All parameters required for successful control are subject of measuring. For example, for the control of the electric motor speed it is enough to measure the speed and supply voltage only, however, the result will be better if there will be measured the machine current as well, and it will be regulated. Not always is the measurement of input necessary, for instance, if it is constant or is changing according to certain regularity. In addition, it could not always be possible to measure the input, especially by some disturbances.

The measuring device is the most widespread element of the automatic control systems, so as there are many quantities, which could be and are to be measured. For instance, in case of an electric drive the following sensors could be possible: - Hall transducer and magnetic resistances for the measurement of the magnetic field of the rotor air gap. tacho-generator or velocity transducer for the measurement of the speed, resolver, optical incremental sensor. The two last suit the determination of the rotor shaft position. For the measurement of the current a current transformer will be suitable, The there will be another shunt and Hall sensor, dynamometer for the measuring of the torque which is usually replaced by the measurement of the current, so as the torque is a product of the current. For the measurement of the voltage transformers, voltage dividers etc are suitable. The question is not about hot or what to measure, but about the purposefulness of the measurement. This problems arises with the processing duration, so as a certain time will be required for the data collection, evaluation and decision or the formation of the control action, therefore, by the time of formation of the control action the system could be in emergency state already.

The measuring devices could be abandoned at all, in case if the output of the control object will be precisely determined in each time moment. One of those control objects could be the step-motor, which has precisely determined rotor position, if the motor is not overloaded.

### 1.3.4.1. Transducers of speed and position

**Tacho-generators** represent a direct current machine, which transforms the rotation speed or mechanical signal into voltage or an electrical signal. [4]. It is described by the following equation

$$u(t) = k_m \cdot \omega(t) \ , \qquad (1.8)$$

where $k_m$ – a machine constant, characterising construction and excitation of the machine.

Main shortcoming of the tacho-generator is the wear of brushes, therefore they require frequent maintenance, and also it is impossible to determine with it the position of the shaft.

There exist asynchronous and synchronous tacho generators, however, they are not widespread

**Resolver** represents a high frequency dynamotor, the primary winding of which is settled on the shaft of the rotating machine and the secondary side has two windings, which are mutually shifted by 90 degrees. With the rotation of the primary winding a signal of variable amplitude is induced on the secondary side, which, after the filtering of the currier frequency gives a sinus-signal A and a co sinus-signal B. Based on the instantaneous values of these signals it is possible to determine the position of the shaft and from that, after differentiation - the speed. The advantage of this device is the absence of brushes [5].

.

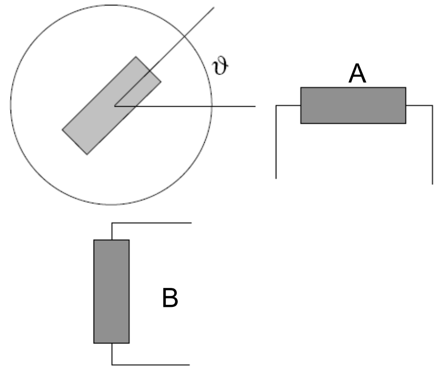
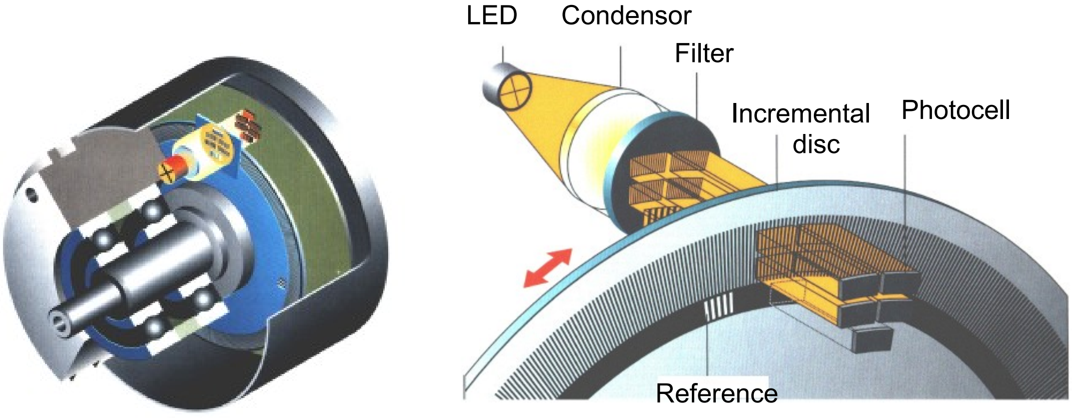



Figure 1.16. Principal diagram of a resolver

Main element of an **optical incremental transducer** (Figure 1.17) is the incremental disc, which is sat on the rotating shaft. The disc is divided into several sectors, light penetrable, and light impervious (1024, 2048 or 4097 light penetrable sectors). These sectors are exposed with LED light sources mainly; the light spot is directed through condenser and filter. The condenser concentrates the light spot and the filter also has light penetrating and light impervious sectors, the width of which corresponds to the width of the disc sectors. On the other side of the disc four photocells are placed, which are shifted in relation to each other be the quarter of the sectors width and which are pair wise switched back-to-back. As a result of it, a situation arises where by the rotation of the incremental disc a current is generated in the photocells what corresponds to the sinus and cosines signals, where the period of the signal corresponds to two sectors width (one penetrating and another impervious), but not to one shaft rotation. Therefore, the frequency of the signals obtained is a product of the number of sectors and of the rotation speed of the shaft. But - with the tangent it is possible to determine the position of te shaft inside of one period, therefore the given transducer has to count how many periods is passed during this turn, or  - it has to increment. To enable the intermediate control of the calculation results the incremental disc has a reference point, which is monitored by the fifth photocell, which fixes, that one turn of the shaft has occurred [5].

There exist another optical position transducer, called absolute transducer or encoder. In this case, the disc is not divided into ribbed arches but into zones (similar to the chess board). The arrangement of the zones depends on the way of the encoding and on the number of bits used in the encoding. As a disadvantages of this transducer are accuracy (the zones could not be placer with the same density as could be placed sectors) and large amount of data (for each bit there exists one photocell [5].




a) b)

Figure 1.17. Optical incremental transducer a) construction b) functioning principle

### 1.3.4.2. Current transducers

**Current transformers** are the most widespread devices for measurement of the current. So one has to deal with a transformer, although with transformers of special type, they are suitable for measuring in the alternating current systems only. The advantage of current transformers is heir simplicity that is expressed in their construction. The primary winding of the transformer is a cord or cable, the current of which is measured, and secondary winding is placed around tha cable and on its terminal an emf, proportional to the current, will be induced

Figure 1.18. Current transformer

**Transducer with Hall's sensor** is settable for operation both by direct current as well by alternating current. The current to be measured crosses the transducer's winding and induces in it a magnetic flux, which crosses the Hall's sensor. So as the magnetic flux is proportional to the magnetic flux. and the voltage on the Hall's sensor terminal is proportional to the magnetic flux, than

Figure 1.19.Hall's transducer

as a result we have a voltage signal proportional to the current [5].

## 1.4. Control methods

**The classical control method** belongs to older and therefore simpler systems. It is used by systems with one input and one output mainly, In this case one has to deal with linear systems which could be described by linear differential equations and which could be analysed by Lap lace' transformation and frequency methods. These systems could in turn be divided into stabilisation, regulating and following systems. **Stabilisation systems** are used in case if it is desired to keep the output on certain given level. **Regulating systems** are applied in case if the output will be changed by some certain principle. **Programmed control systems** belong to this category. **Following** systems change their output in dependence on their input, the changes of which are not planned beforehand.

*Example 1.2.*

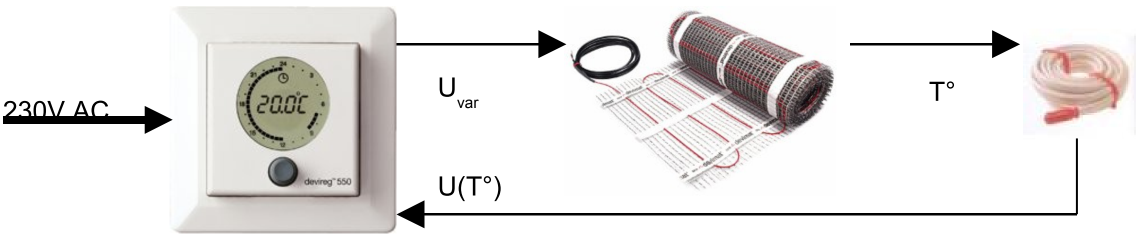



Figure 1.21. Automatic control system with a classical control method

Required room temperature is set on the thermostat, Depending on the difference of the actual and desired room temperatures the regulator builds up, by the mean of the electronics of supply block, a voltage suitable for supply cable, which in turn determines the heating intensity.

The temperature is measured by a transducer, which transforms the temperature into voltage signal, which is directed to the regulator.

**Modern control methods are used by multicoupled systems,** i.e. by systems with multiple inputs and outputs, where multiple outputs are regulated simultaneously.

*Example 1.3.*

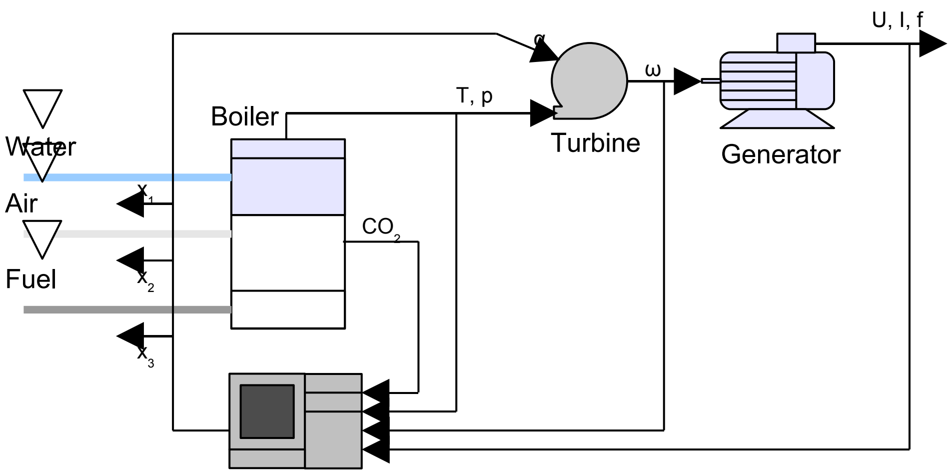


Figure1.22. Automatic control system with modern control method

In the figure 1.22, an electricity production complex is represented in a simplified way. Into the combustion chamber air and fuel are sent, emerging energy in the combustion of which is transferred to the water steam, which rotates the turbine In this connection the generator also will rotate and electricity will be produced. To hold the network voltage and frequency by changing load stable, the load current will be monitored. Therefore the rotation speed of the generator has to be regulated, which is made or by changing the angels of turbine blades or changing of the steam energy. The latter could be changed by changing the pressure and temperature of the steam, which is made by boilers pump (not shown in the figure) or changing the combustion process adding fuel and air to it as required. During the combustion process, the exhaust gases are monitored, to provide the optimal combustion process and maximal efficiency.

Description of such systems by means of transfer functions would be complicated; therefore, state equations in digital form are mainly used for the description of modern control systems.

The highest of control methods are the intellectual methods, which are based on the intuitive estimations of the programmer, on the fuzzy logic or expert evaluation, for instance. These methods are applied when one has to deal with the uncertainty of the control object. In this case, the state variables are not estimated quantitatively but qualitatively (large, small, for example) and these variables will be related to the conditional clauses IF-THEN according to the algorithm. To such systems self-adjusting, self-programming, self-learning and self-organising systems belong [2]

## 1.5. The purpose of the control

Proceeding from the given above the purpose of the control could be formulated as follows [2]:

The main purpose of the control is to form a set of control actions, in result of which the system will behave as desired. This set of actions has to provide to he system a control corresponding to the quality requirements (reaction speed, overshoot, attenuation) of the system. Thereby, one has to consider the physical conditions (power limits of the equipment) and subjective decisions (danger to the personnel).

Solution of this problem means the synthesis or creation of the automatic control system. Therefore, the object under study will be separated from the surrounding media, and its parameters and structure will be determinate. This is called **identification of the object.** Quit often it occurs notional, so as in the project stage the does not exist a real object where from a mathematical **model** imitating the behaviour of a real object could be composed. If the real object does exist then it is possible to identify it investigating the reactions of the object or transients produced by standardized input signals. . That reaction could be also called as signal echoes. Mathematical equations or models could describe the shape of the signal echo.

How well the model behaves, depends on that how exactly is the control object modelled. The more complicated is the object the greater is the rate of uncertainty. Therefore, as the uncertainty is described by the evaluations, the uncertainties are arising into model, in turn complicating the solution of the problem. In case if the model is composed without the presence of any indeterminacy, the model is called fully determinate. Those systems could be easily controlled by open loop, however, in practice, such systems are rather exclusions, therefore most of real systems operate on the feedback principle, to diminish the system with the application of measuring devices and the uncertainty of the model with it.

Most of the mathematical models and of the systems are **dynamic**, i.e. the state of the system depends on the previous one. For instance, the movement of a link of the serial cinematic robot is influenced by the movements of all other links. Those systems could be called **inertial systems** or **systems with after-effect**. These systems are described by the means of differential or difference equations.

As next stage, the objective of the control will be set. Therefore, an output variable will be determined which will be controlled, deflecting torque, angle velocity or position for instance, in case of an electric motor, and for instance, the torque will be hold constant or will be changed according to certain law.

In the run of the solution of a control problem the permissible set signals and control actions. These signals must be determined beforehand due to the limitation formulated in the task.

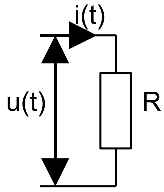
As last stage, the measure of the control quality must be determined. For this purpose a set of different criteria serve, based on which it is possible to compare different systems, executing the same tasks. Also it is possible based on this measure to optimise the system, evaluate the system operation and reduce the influence of disturbances to the output.

# 2. DESCRIPTION OF AUTOMATIC CONTRL SYSTEMS

For sufficient control of an automatic control system the dynamic properties of both - as of the process as well of the system, must be known. The dynamic properties of a component or of a system could be determined by calculations or experimentally.

## 2.1. Description of systems by differential equations

*Example2.1.*




For the resistor Ohm's law holds

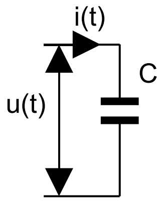
$$u(t)=R\cdot i(t)\ .$$

So as the resistance is depending on the temperature, then more precisely will it be

$$u(t)=R(i,t)\cdot i(t)\ .$$

Fidure2.1.A resistor

*Example 2.2.*

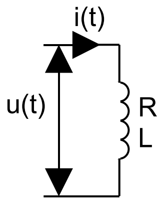



Voltage equation of a condenser

$$u(t)=\frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau=U_0+\frac{1}{C}\int_{0}^{t} i(\tau)\,d\tau\ .$$

Figure 2.2. A condenser

*Example 2.3.*




The voltage drop of a winding is expressed as follows

$$u(t)=R\cdot i(t)+L\cdot\frac{di(t)}{dt}\ .$$

Figure 2.3. A coil

*Example 2.4.*

Let it be a dripping buckle with water hanged on the spring [3]. Proceeding from the Newton's II law and Hook's law for this system could be written

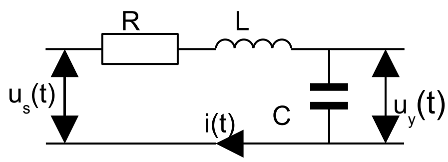
$$\frac{d}{dt}[m(t)\cdot\dot{x}]+k(x)\cdot x=m(t)\cdot g$$

$$m(t)\cdot\ddot{x}+\dot{m}(t)\cdot\dot{x}+k(x)\cdot x=m(t)\cdot g$$

Joonis 2.4. Lekkiv anum

*Example 2.5.*

For a RLC circuit the Kirchof-s second law and corresponding differential equations could be applied.



$$u_s(t) = R \cdot i(t) + L \cdot \frac{d\,i(t)}{dt} + u_y(t)$$

$$i(t) = C \cdot \frac{d\,u_y(t)}{dt}$$

$$u_s(t) = RC \cdot \frac{d\,u_y(t)}{dt} + LC \cdot \frac{d^2 u_y(t)}{dt^2} + u_y(t)$$

Figure 2.5. RLC-C-circuit

*Example 2.6.*

Based on Newton\s II law the following is valid>

$$m \cdot \frac{d^2 x}{dt^2} = F - \mu \cdot m \cdot g \cdot (sgn \frac{dx}{dt})$$

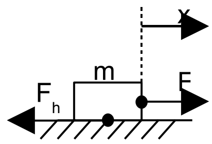



Figure 2.6. Linear motion

*Example 2.7.*

Newton's II law holds for the rotating movement also

$$J \frac{d\,\omega}{dt} = T - T_h(\omega) \ .$$

The difference here is, that frictional moment always works against of the drive moment

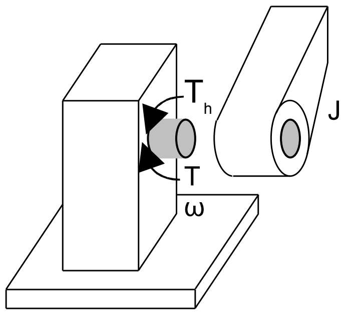
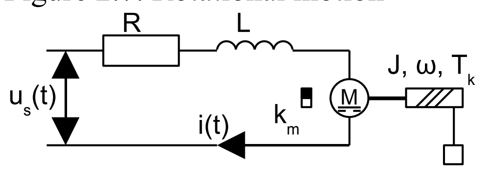



Figure 2.7. Rotational motion

*Example 2.8.*

From the theory of electrical drives it is known the fundamental equation of the direct current electric motor:




$$u_s(t) = R \cdot i(t) + L \cdot \frac{d\,i(t)}{dt} + k_m \cdot \omega(t)$$

$$J \cdot \frac{d\,\omega(t)}{dt} = k_m \cdot i(t) - T_k(t)$$

Figure 2.8. A DC drive

Assuming, that inertial moment and the load moment of the system are constant in time (the frictional moment is negligible compared to the load moment), then it is possible to replace mechanical equation into voltage equation:

$$u_s(t) = \frac{J \cdot L}{k_m} \cdot \frac{d^2 \omega(t)}{dt^2} + \frac{J \cdot R}{k_m} \cdot \frac{d\,\omega(t)}{dt} + k_m \omega(t) + \frac{R}{k_m} \cdot T_k(t) + \frac{L}{k_m} \cdot \frac{d\,T_k(t)}{dt} \ .$$

## 2.2. Description of systems by the mean of transfer functions

### 2.2.1. Laplace transformation

Solution of differential equations without support of mathematical software is complicated, therefore there are various solution techniques are developed for them. One of the oldest is the operator method or Laplace transformation.

$$F(p)=\mathscr{L}\{f(t)\}=\int_0^{\infty} f(t)\cdot \mathrm{e}^{-p\cdot t} , \tag{2.1}$$

where F(p) – operator representation of the differential equation
*f* (t) – differential equation
p – complex variable or operator

Here one has to deal with a mathematical manipulation, in the run of which the time axis of the process will be turned around or switched over to the frequency axis. As a result of it processes that had occurred in the beginning of the time axis or fast processes are imaged at the end of frequency axis and slow processes at the beginning of frequency axis. The mathematical background of such manipulation is not easy to understand, however, solution of differential equation in operator form is possible not knowing the integration of differential equations, so as the differential equations will be replaced with simple algebraic functions.

The execution of transformations is comparatively easy, however, the inverse transformation or returning into time space is a complicated mathematical procedure, therefore for the execution of the transformations tables (Appendix 1) are used, if there do not exist ready software solutions at hand.

Transformation of a differential

$$\mathscr{L}\left\{\frac{d^n}{dt^n} f(t)\right\}=p^n\cdot F(p) \tag{2.2}$$

And on an integral

$$\mathscr{L}\left\{\int f(t)\right\}=\frac{1}{p}\cdot F(p) . \tag{2.3}$$

*Example 2.9.*

The representation function of the differential equation of the Example 2.9.

$$u(t)=R\cdot i(t)+L\cdot\frac{di(t)}{dt} \;\circ\!\!-\!\!\bullet\; u(p)=R\cdot i(p)+p\cdot L\cdot i(p)$$

*Example 2.10.*

The representation function of the differential equation of the Example 2.10

$$u_s(t)=\frac{J\cdot L}{k_m}\cdot\frac{d^2\omega(t)}{dt^2}+\frac{J\cdot R}{k_m}\cdot\frac{d\,\omega(t)}{dt}+k_m\omega(t)+\frac{R}{k_m}\cdot T_k+\frac{L}{k_m}\cdot\frac{d\,T_k(t)}{dt}\;\circ\!\!-\!\!\bullet$$

$$u_s(p)=\frac{J\cdot L}{k_m}\cdot\omega(p)\cdot p^2+\frac{J\cdot R}{k_m}\cdot\omega(p)\cdot p+k_m\cdot\omega(p)+\frac{R}{k_m}\cdot T_k(p)+\frac{L}{k_m}\cdot p\cdot T_k(p)$$

## 2.2.2. The transfer function

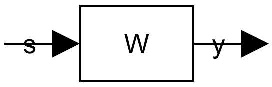



Figure 2.9. The transfer link

The transfer function describes the relation of the output variable of a transfer link to the input variable of it.

$$W=\frac{y}{s} \; . \tag{2.4}$$

The transfer function will be transformed from the representation function due to its simplicity. One of the advantages of the transfer function is its reversibility

$$W^{-1}=\frac{s}{y} \; . \tag{2.5}$$

*Example2.11.*

In the Example 2.3 a RL cit was presented and in the Example 2.9 the representation function of it was given. Determining the supply voltage being as input variable and the current as output variable the corresponding transfer function could be found as follows

$$u(p)=R\cdot i(p)+p\cdot L\cdot i(p)=[R+p\cdot L]\cdot i(p)$$

$$W(p)=\frac{i(p)}{u(p)}=\frac{1}{R+p\cdot L}=\frac{\frac{1}{R}}{1+p\cdot\frac{L}{R}} \; .$$

The transfer function found is a little bit unhandy, it is recommended to do some more transformations

$\frac{1}{R}=K$ and $\frac{L}{R}=T$ ,

where K – transfer ratio;
T – time constant of the process

$$W(p)=\frac{i(p)}{u(p)}=\frac{K}{1+p\cdot T}$$

The transfer ratio describes the gain of given link and it is always a value with a dimension. (In given case $\frac{1}{\Omega}=\frac{A}{V}$ ), the unit of which determines the unit of the transfer function

The time constant of the process describes the temporal properties of the transient processes occurring in the transfer link. This is also a value with a dimension, the unit of which is a second. So as the transfer function determined the transfer function unit, the it follows from here, that the denominator of the function must be without dimension, from where it follows in turn that operator p also is a value with a dimension, the unit of which is $s^{-1}$.

If to consider the current as input variable and voltage as output variable, then the corresponding transfer function will be

$$W(p)=\frac{u(p)}{i(p)}=R+p\cdot L=\frac{1+p\cdot T}{K} .$$

The result is reciprocal of the first determined transfer function, which reduces essentially the amount of calculations in the analysis and synthesis of automatic control systems

*Example 2.12.*

In the Example 2.5 a RLC circuit was considered the output variable was the condensers voltage and input variable the supply voltage. Corresponding to this transfer function will be determined as follows

$$u_s(t)=RC\cdot\frac{d\,u_y(t)}{dt}+LC\cdot\frac{d^2u_y(t)}{dt^2}+u_y(t) \;\multimap\; u_s(p)=RC\cdot p\cdot u_y(p)+LC\cdot p^2\cdot u_y(p)+u_y(p)$$

$$W(p)=\frac{u_y(p)}{u_s(p)}=\frac{1}{LC\cdot p^2+RC\cdot p+1} .$$

Substituting $RC=T_2$ and $\frac{L}{R}=T_1$ the transfer function will have the form

$$W(p)=\frac{u_y(p)}{u_s(p)}=\frac{K}{T_1T_2\cdot p^2+T_2\cdot p+1} .$$

*Example 2.13.*

In the Example 2.8 the representation function of a DC drive was brought

$$u_s(p)=\frac{J\cdot L}{k_m}\cdot\omega(p)\cdot p^2+\frac{J\cdot R}{k_m}\cdot\omega(p)\cdot p+k_m\cdot\omega(p)+\frac{R}{k_m}\cdot T_k(p)+\frac{L}{k_m}\cdot p\cdot T_k(p) .$$

In given function there are three non-parametric values $u_s$, $\omega$ and $T_k$. The transfer function presumes two non-parametric variables only. So as the electrical drive is controlled by voltage. then it could be considered as input variable. However, load moment is a random value which is not possible to control, and therefore could be considered rather as a disturbance then a control action or output variable. The velocity thereby is easy to control and eve easier to measure (with a tachogenerator)

Therefore it suits for an output variable. Thus, we have to deal with one output value and two input values (disturbance also is an input variable), hence – two transfer functions could be expressed:

$$W_j(p)=\frac{\omega(p)}{u_s(p)}=\frac{1}{\frac{J\cdot L}{k_m}\cdot p^2+\frac{J\cdot R}{k_m}\cdot p+k_m} ,$$

$$W_h(p)=\frac{\omega(p)}{T_k(p)}=\frac{-\frac{R+L\cdot p}{k_m}}{\frac{J\cdot L}{k_m}\cdot p^2+\frac{J\cdot R}{k_m}\cdot p+k_m} .$$

After arrangement and making substitutions $\frac{1}{k_m}=K$ , $\frac{J\cdot R}{k_m^2}=T_2$ and $\frac{L}{R}=T_1$ we will have in result:

$$W_j(p)=\frac{\omega(p)}{u_s(p)}=\frac{K}{T_1 T_2\cdot p^2+T_2\cdot p+1},$$

$$W_h(p)=\frac{\omega(p)}{T_k(p)}=\frac{-\frac{R+L\cdot p}{k_m^2}}{T_1 T_2\cdot p^2+T_2\cdot p+1}.$$

These are called transfer function and disturbance transfer functions correspondingly. Each of them describes the dependence of the output on the different input signal and as one can see the transfer functions are different, and as a prevision it could be mentioned that if the drive is adjusted on the ideal execution of the control then the reduction of the disturbances influence is less effective and the same holds to the opposite case. We shall return to this problem in the following chapters.

If to compare the transfer functions of the RLC circuit and of the DC drive, one could be convinced that mathematically we are dealing with similar systems, although their physical backgrounds are different. From this one my conclude, that systems behave similarly, whereas they could be described similarly and controlled similarly or, typical solutions could applied to them and they could be named standard links. Standard links are considered in Chapter 2.5.

### 2.2.3. The Bode diagram

Bode diagram is otherwise called logarithmic amplitude-phase frequency characteristic. Here one has to deal with graphical representation of the transfer function, characterising the dependence of system properties on the frequency. It is widely used in the acoustics and by the calculations of filters, thou as it is possible to see on them whether the given object by given frequencies gives a damping or amplification of the signal, and which will be the phase shift of the signal. In the automatic control we don't deal with continually changing frequency processes, however, for the analysis of the systems the Bode diagram is useful, so as it is easy to determine about system stability on the diagram (see also Chapter 3.1)

There are two different representation forms of Bode diagram, but historically it has developed here in Estonia the use of the representation, where decibel (dB) is taken as measure of logarithmic amplitude, which is used as a relative gain. Also the application of per units enables to join different objects graphically.

While describing the relation of the input and output power of an object the logarithmic mode of the description is used which simplifies their representation as graphics. Mathematically it is expressed as

$$L=\log\frac{P_v}{P_s} \tag{2.6}$$

and the unit is 1 bell, that means the relation of powers 10:1. So as quite often we have to deal with smaller relations, the as a unit *decibel* is used

$$L=10\cdot\log\frac{P_v}{P_s} \tag{2.7}$$

The unit (decibell was at the beginning related to the power or „squared values“), then for the linear values based on in the electricity known relation $P \sim I^2$ holds

$$L = 10\cdot\log\frac{P_v}{P_s} = 10\cdot\log\frac{I_v^2}{I_s^2} = 20\cdot\log\frac{I_v}{I_s} = 20\cdot\log K \ . \quad (2.8)$$

The transformation of the transfer function to the logarithmic form is made by the condition that p = jω

$$\log W(p) = \log W(j\omega) = \log K(\omega)e^{j\varphi(\omega)} = \log K(\omega) + j\varphi(\omega)\ , \quad (2.10)$$

However, if the Bode diagram is constructed by hand, then it will be solved by asymptotes

*Example 2.9.*

Let be given a transfer function

$$W(p) = \frac{K\cdot(1+p\cdot T_1)}{1+p\cdot T_2}\ ,$$

From which the frequency function will be expressed

$$W(j\omega) = \frac{K\cdot(1+j\omega\cdot T_1)}{1+j\omega\cdot T_2}\ .$$

The frequency function could be divided into two separate functions

$W_1(j\omega) = 1 + j\omega\cdot T_1$ and $W_2(j\omega) = \dfrac{K}{1+j\omega\cdot T_2}$ .

For this functions the corner frequencies will be determined:

$\omega_1 = \dfrac{1}{T_1}$ and $\omega_2 = \dfrac{1}{T_2}$ .

Proceeding from the corner frequencies the asymptotes of both of functions determinate

$$W_1(j\omega) = 1 + T_1 j\omega|_{\omega\ll\omega_1} = 1$$
$$W_1(j\omega) = 1 + T_1 j\omega|_{\omega\gg\omega_1} = T_1(j\omega)^1\ ,$$

$$W_2(j\omega) = \frac{K}{1+j\omega\cdot T_2}|_{\omega\ll\omega_2} = K$$
$$W_2(j\omega) = \frac{K}{1+j\omega\cdot T_2}|_{\omega\gg\omega_2} = \frac{K}{T_2}(j\omega)^{-1}\ .$$

The value of the power of the frequency variable is showing the inclination of the asymptote. In case of positive power the asymptote proceeds up and by negative values it proceeds down. The value of the power determines the value of the inclination (slop), or how many amplitude decades (20 dB) will the amplitude change during one frequency decade (10x). Also by the value of power are phase shifts determined. – the sign shows the direction of the phase shift and its value shows how many 90-degree phase shifts occurs.

Let it be given for the construction of a diagram K = 2, $T_1$ = 2 s and $T_2$ = 0,5 s.

With it, the corner frequencies are

$\omega_1 = \frac{1}{2\,s} = 0{,}5\ s^{-1}$ and $\omega_1 = \frac{1}{0{,}5\,s} = 2\ s^{-1}$ .

The asymptotes are expressed as:

$W_{1.1}(j\omega) = 1$ , $W_{1.2}(j\omega) = 2\ s(j\omega)$ , $W_{2.1}(j\omega) = 2$ and $W_{2.2}(j\omega) = 1\ s^{-1}(j\omega)^{-1}$ .

The logarithms of the gains are

$L_{1.1} = 20 \cdot \log 1 = 0$ and $L_{2.1} = 20 \cdot \log 2 = 6{,}02$ .

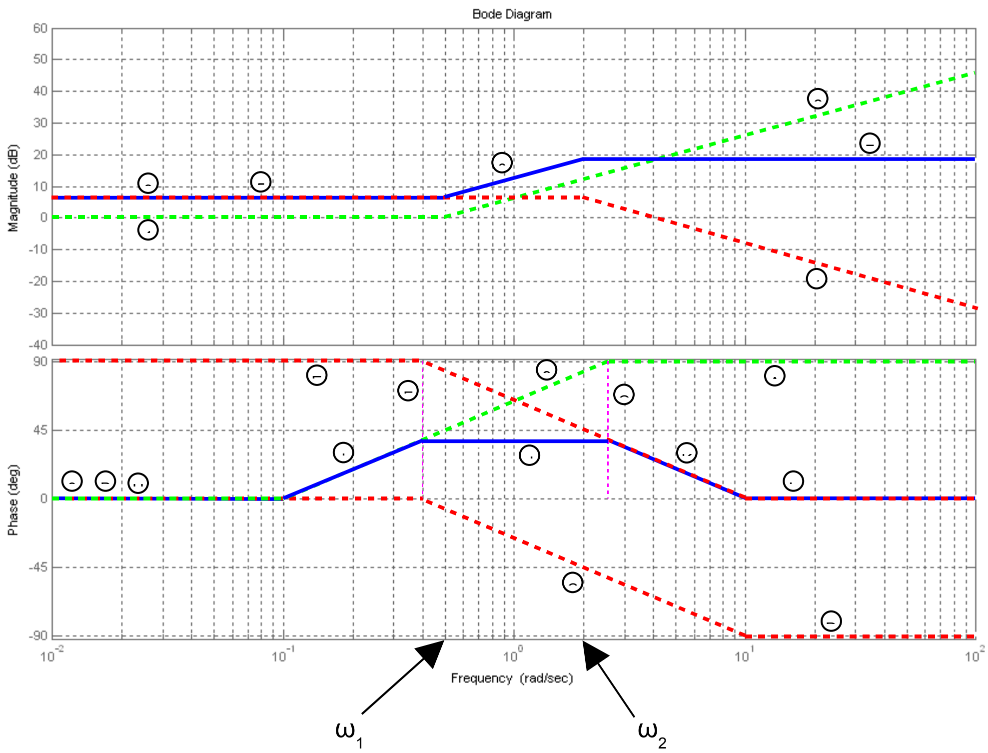



Figure 2.10. The asymptotes of the Bode diagram

By the construction of the Bode diagram one proceeds from the following preceding calculations

| | |
|---|---|
| Step 1 | The gain of the first transfer function is 0 dB until the first corner function |
| Step 2 | After the corner frequency the gain will increase with the rate 20 dB for one decade |
| Step 3 | The gain of the other transfer function is 6,02 dB until the next corner frequency |
| Step 4 | After the corner frequency the gain will decrease with the speed 20 dB for a decade |
| Step 5 | The gains of different functions could be arithmetically added, so the gain of the system is 6,02 dB until the first corner frequency |
| Step 6 | After first corner frequency the gain of the first function will increase also, and therefore the gain of the whole system with a speed 20 dB for a decade |
| Step 7 | After the second corner frequency the gain of the second function starts to decrease and so as the velocities of the increase and of the decrease are equal, the gain of the system does not increase nor decrease.. |
| Step 8 | At low frequencies the phase shift of the first subsystem is missing |
| Step 9 | In the first subsystem at the corner frequency the phase shift occurs in the positive direction, for construction of which a point will be marked, the coordinates of it are corner frequency and half of the final value of the phase shift (in given case 45 degrees). Through this pint a line will be drown, which begins in the point with coordinates initial phase angel and 0.7 decade length before the corner frequency and ends in the point the |

coordinates of which are final phase angel and 0.7 of the decade length after the corner frequency.

Step A The first subsystem does not have any more corner frequencies, therefore it holds the phase angel obtained at the end of the first step

Step B The second subsystem also begins with a 0-degree phase shift.

Step C In the given subsystem the phase shift occurs in the negative direction, which will be constructed similarly to the step 9

Step D In this system also the other corner frequencies are missing

Step E So as the logarithmic systems could be added, then for the sake of simplification of the mathematics the phase-frequency characteristic line of the second subsystem will be shifted by 90-degrees upwards.

Step F The phase shifting processes are partly overlapping and a vertical line gives the lower limit of it by frequency.

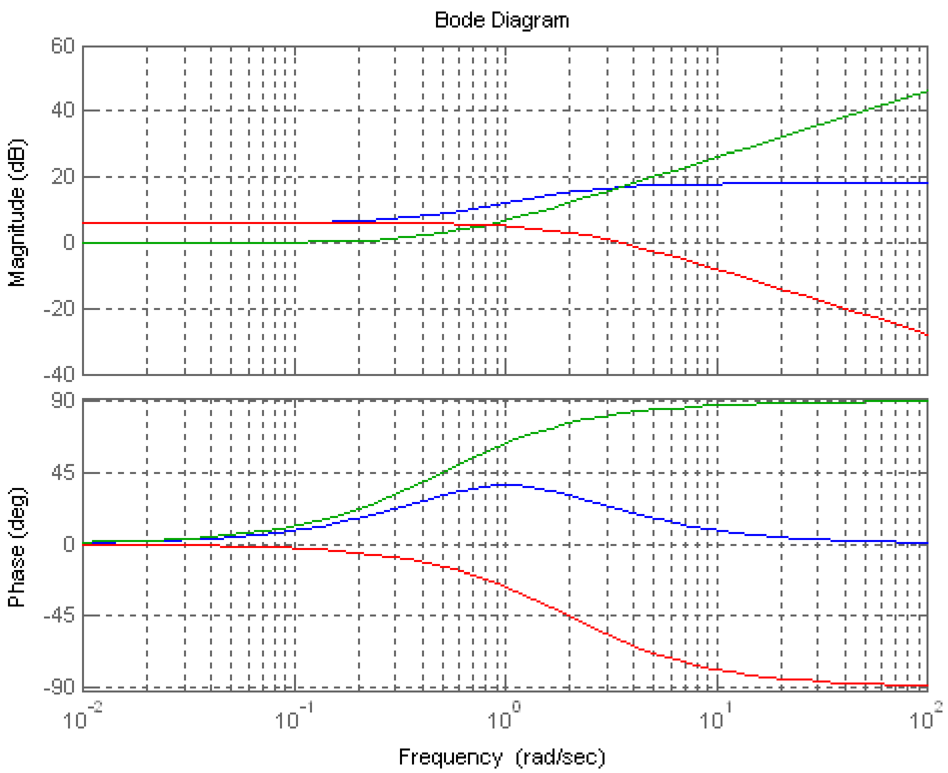
Step G Similar to the step F, describes the determination of the upper limit

Step H Adding two systems we will get as a result overall phase shift equal zero to the first phase shifting process

Step I The first subsystem produces a positive phase shift until the lower limit of overlapping

Step J Between the limits of overlapping the slopes of both processes are equal but opposite signs, therefore the system keeps the phase shift obtained by the previous step.

Step K After the upper limit of overlapping the whole system is influenced by the second subsystem, producing negative phase shift




Figures 2.11. The calculated Bode diagram

## .2.4. Structure diagrams

In case if the system appears to be too complicated, it will be presented consisting from separate transfer functions, which could simplify the understanding of the processes occurring in the system.

*Example 2.10.*

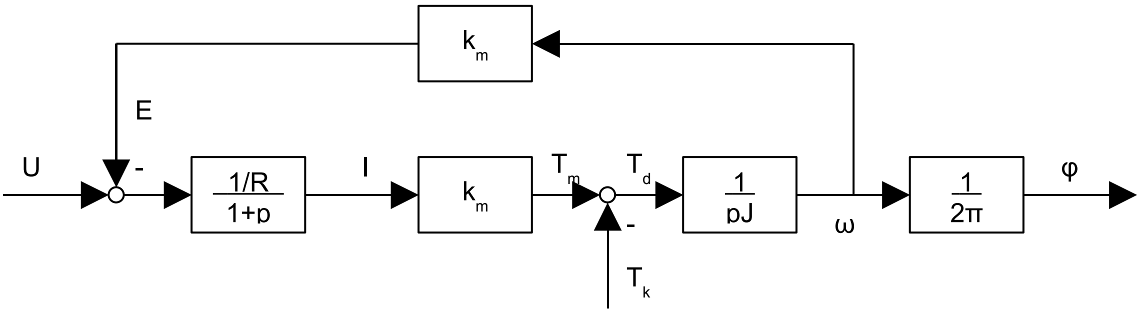



Figure 2.12. Structure diagram of a separately exited DC motor

The first transfer function described transformation of the voltage signal into a current signal, which in turn will be transformed into machine torque. The difference of the machine torque and load torque is a dynamic moment, by division of it by inertial torque and integrating, one will have the speed signal. Integrating the speed signal in turn one gets as a result the inductors position [6]

And so as we have to deal with simple structures, the transfer function of the control could be expressed by the following

$$W_j(p)=\frac{\omega(p)}{U(p)}=\frac{\dfrac{\frac{1}{R}}{1+p\frac{L}{R}}\cdot k_m\cdot\dfrac{1}{pJ}}{1+\dfrac{\frac{1}{R}}{1+p\frac{L}{R}}\cdot k_m\cdot\dfrac{1}{pJ}\cdot k_m},$$

Which, after arrangement and substitutions reduces to the form

$$W_j(p)=\frac{\omega(p)}{U(p)}=\frac{K}{T_1T_2\cdot p^2+T_2\cdot p+1}.$$

The same transfer function was determined in the Example 2.8.

In case if the automatic control system is not presented by simple structure diagrams, then, for the determination of the transfer function of this system the following possibilities exist:

- transform the structure diagram into simple schemes
- represent the structure diagram with open loops
- to use the Mason's equation

The mentioned above solution methods are analysed by determination of the transfer function of the automatic control system presented in the figure 2.13. [2].

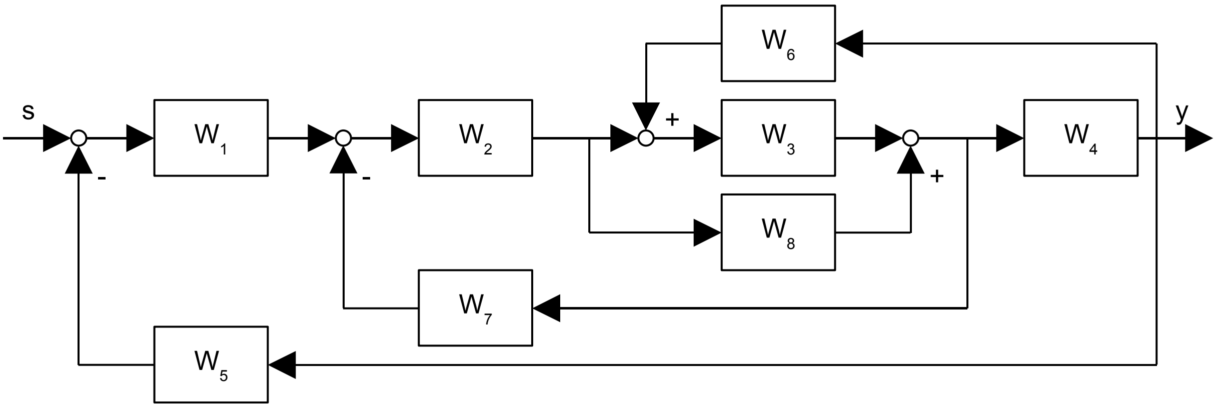



Figure 2.13. Structure diagram of an automatic control system

### 2.2.4.1. Determination of the transfer function by transformations

For the determination of the transfer function of a structure diagram by transformations one has to begin from the middle of the structure diagram, where it is possible to determine a simple part of the structure diagram, for which it is possible to determine unique one input and one output. In the first chapter simpler structure diagrams were given.

As a middle point of the automatic control system presented in the figure 2.13 could be considered the transfer link $W_3$, which builds up a parallel structure with the link $W_8$, if there wa[illegible] adding link in the input of the link [illegible] llows from which, that given sc[illegible]es not have simple structures a[illegible] tructure mast be transformed. Transformed could be distribution and addition links from the input to the output of the link and contrariwise, thereby the values of the si[illegible]st remain unchanged after transformation

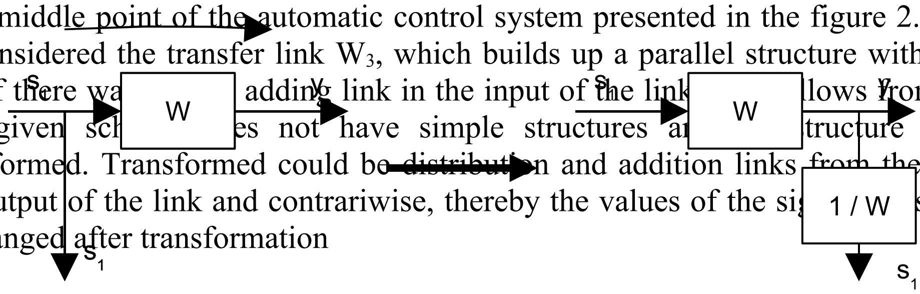


Figure 2.14. Transformation of a distribution link from the input to the output.

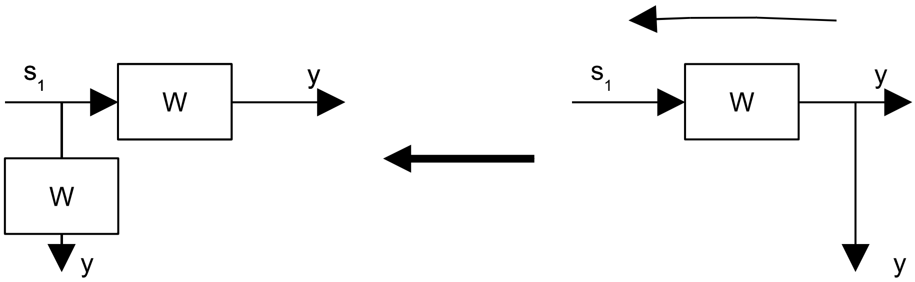



Figure 2.15. Transformation of a distribution link from the output to the input

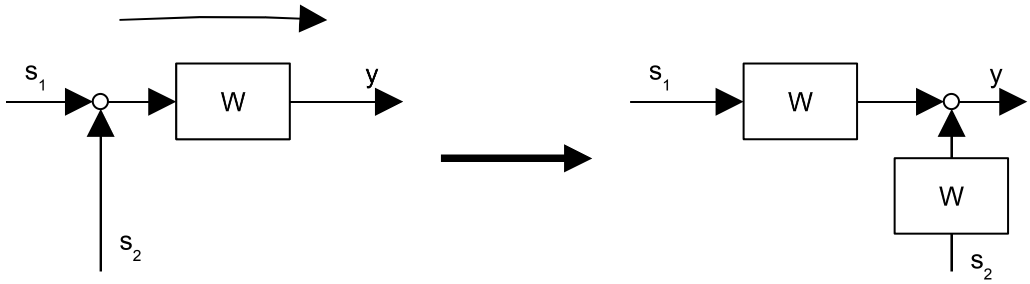



Figure 2.16.Transformation of an addition link form the input to the output

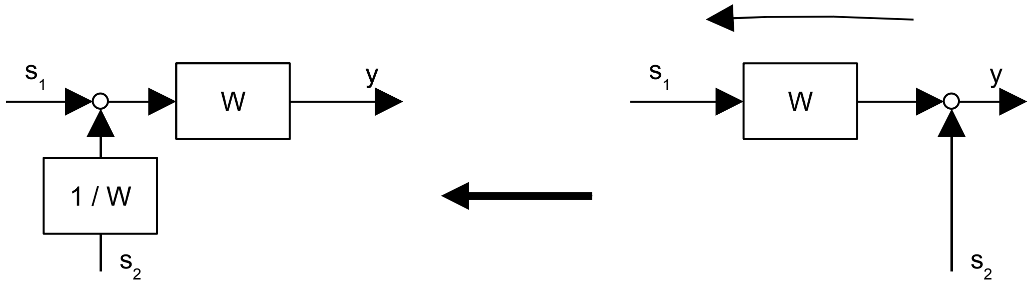



Figure 2.17. Transformation of an addition link from the output to input.

*Example 2.11.*

Preceding from these transformation rules the addition link $W_3$ could be transformed from the input to the output, as a result of what the structure dia[illegible] form:

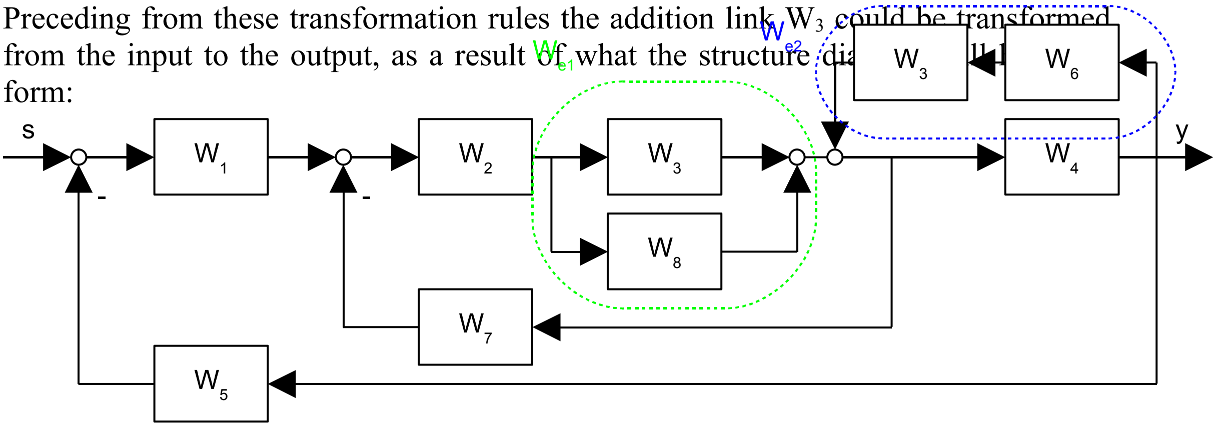



Figure 2.18. The transformed diagram after first step

After the transformation in the structure diagram two simple parts of structure arise, which could be replaced by equivalent transfer links, the transfer function of which are expressed

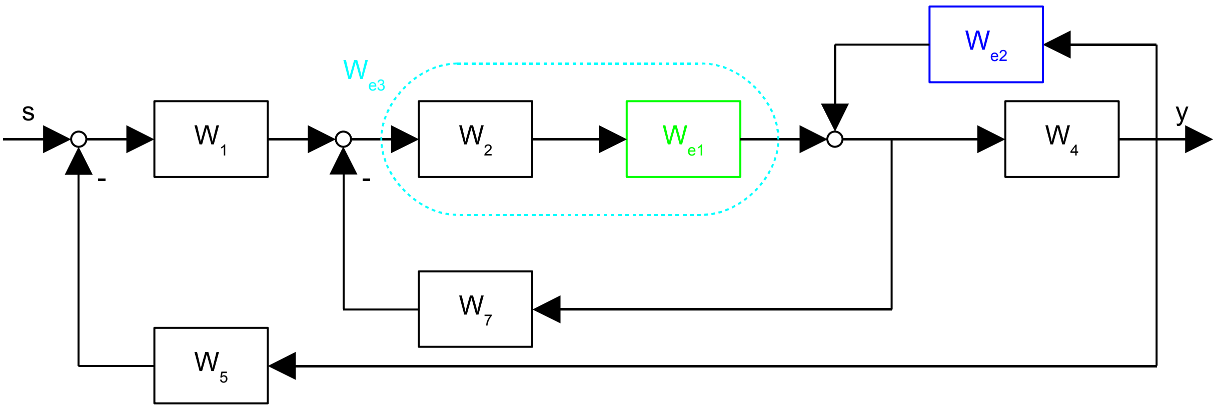
$$W_{e1}=W_3+W_8$$

and

$$W_{e2}=W_3 \cdot W_6$$

and the structure diagram will take the form (Figure 2.19), where a new part of structure diagram arises, the transfer function of which is as follows:

$$W_{e3}=W_{e1} \cdot W_2=(W_3+W_8) \cdot W_2 \ .$$




Figures 2.19. The transformed diagram after second step.

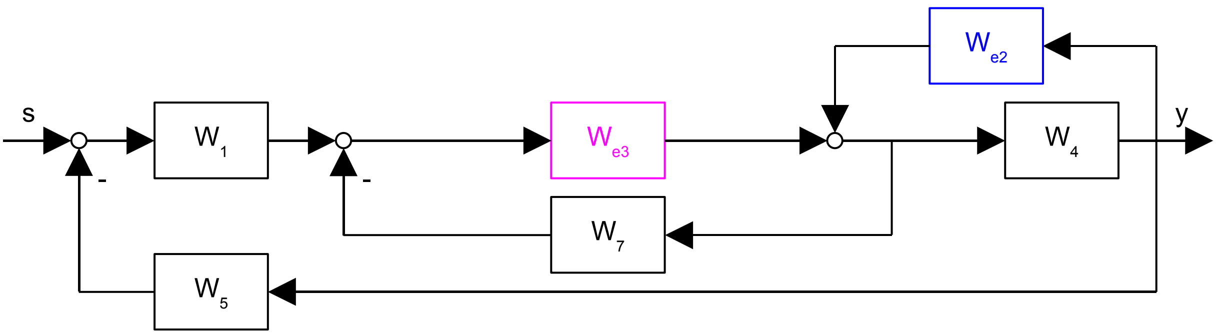



Figure 2.20. The transformed diagram after 3rd step

The following stem will be the transformation of the addition link $W_{e3}$ from the output to input

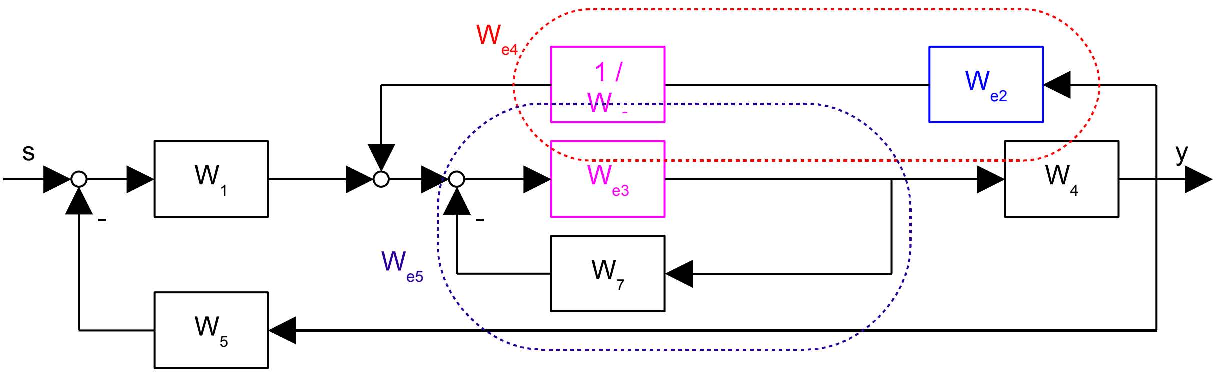



Figure 2.21.The transformed diagram after step 4

After the transformation two following simple structure arose, the transfer functions of which are expressed as

$$W_{e4} = \frac{W_{e2}}{W_{e3}} = \frac{W_3 \cdot W_6}{(W_3 + W_8) \cdot W_2}$$

and

$$W_{e5} = \frac{W_{e3}}{1 + W_{e3} \cdot W_7} = \frac{(W_3 + W_8) \cdot W_2}{1 + (W_3 + W_8) \cdot W_2 \cdot W_7} \; .$$

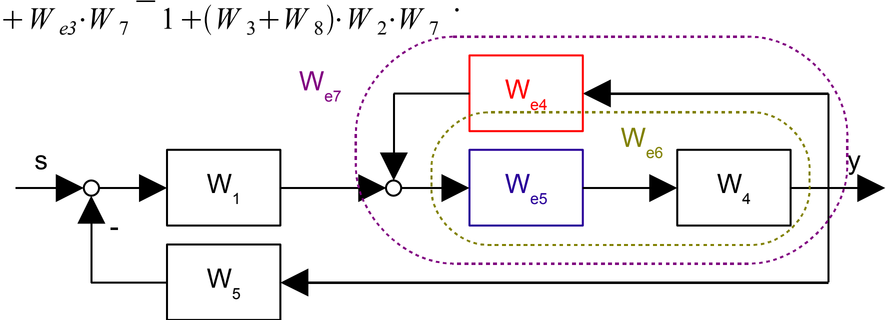


Figure 2.22.The transformed diagram after step 5

The equivalent transfer link of the arising serial structure

$$W_{e6}=W_{e5}\cdot W_4=\frac{(W_3+W_8)\cdot W_2\cdot W_4}{1+(W_3+W_8)\cdot W_2\cdot W_7}$$

builds with the link $W_{e4}$ a feedback structure the equivalent of which will be

$$W_{e7}=\frac{W_{e6}}{1-W_{e4}\cdot W_{e6}}=\frac{\dfrac{(W_3+W_8)\cdot W_2\cdot W_4}{1+(W_3+W_8)\cdot W_2\cdot W_7}}{1-\dfrac{W_3\cdot W_6}{(W_3+W_8)\cdot W_2}\cdot\dfrac{(W_3+W_8)\cdot W_2\cdot W_4}{1+(W_3+W_8)\cdot W_2\cdot W_7}}=$$

$$=\frac{(W_3+W_8)\cdot W_2\cdot W_4}{1+W_2\cdot W_3\cdot W_7+W_2\cdot W_7\cdot W_8-W_3\cdot W_4\cdot W_6}$$

.

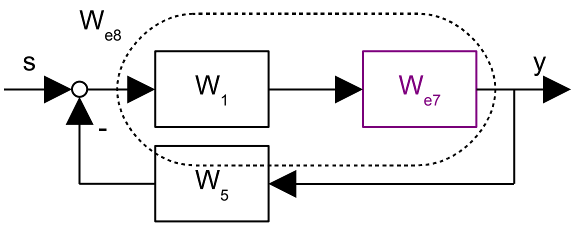



Figure 2.23. The transformed structure after step 7

Next to last transformation will be the transformation of the serial structure

$$W_{e8}=W_1\cdot W_{e7}=\frac{(W_3+W_8)\cdot W_2\cdot W_4\cdot W_1}{1+W_2\cdot W_3\cdot W_7+W_2\cdot W_7\cdot W_8-W_3\cdot W_4\cdot W_6},$$

that forms for the last transformation a feedback structure of the forward transfer

The result of the last transformation is the transfer function of the whole system

$$W=\frac{W_{e8}}{1+W_{e8}\cdot W_5}=\frac{\dfrac{(W_3+W_8)\cdot W_2\cdot W_4\cdot W_1}{1+W_2\cdot W_3\cdot W_7+W_2\cdot W_7\cdot W_8-W_3\cdot W_4\cdot W_6}}{1+\dfrac{(W_3+W_8)\cdot W_2\cdot W_4\cdot W_1}{1+W_2\cdot W_3\cdot W_7+W_2\cdot W_7\cdot W_8-W_3\cdot W_4\cdot W_6}\cdot W_5}=$$

$$=\frac{W_1\cdot W_2\cdot W_3\cdot W4+W_1\cdot W_2\cdot W_4\cdot W_8}{1+W_1\cdot W_2\cdot W_3\cdot W_4\cdot W_5+W_1\cdot W_2\cdot W_4\cdot W_5\cdot W_8+W_2\cdot W_3\cdot W_7+W_2\cdot W_7\cdot W_8-W_3\cdot W_4\cdot W_6}$$

.

### 2.2.4.2. Determination of the transfer function of an open loop structure diagram

*Example 2.12*

This method is efficient for the verification of the transfer function of an automatic control system that is determined by other methods. Switching off the distribution points of diagram represented in figure 2.13 a [illegible] oop structure diagram presented in figure 2.24 is obtained.

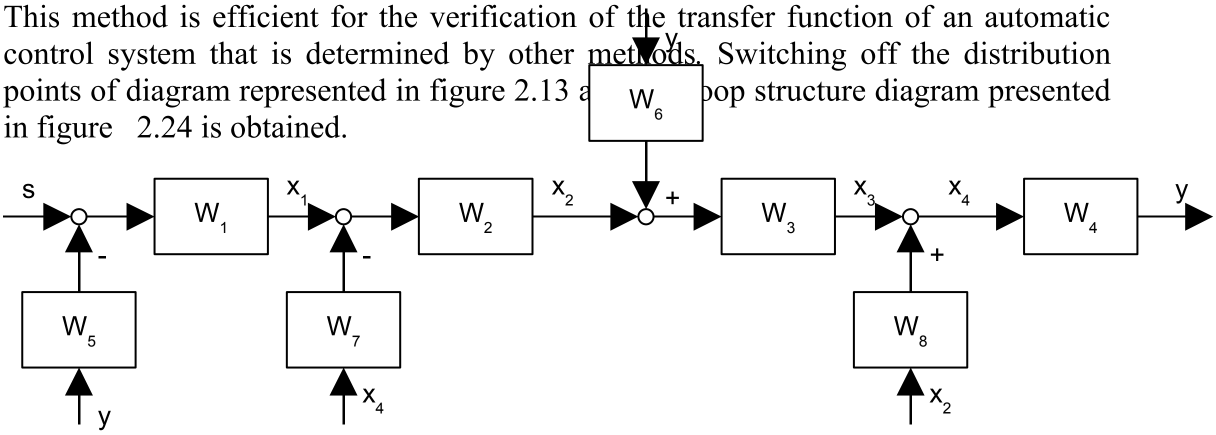



Figure 2.24. Structure diagram with open loops

To simplify the calculations, intermediate variables $x_i$. are added to the system. By the replacement of the intermediate variables it must be taken care, that the number of them is enough for description of the system after splitting of the distribution points. The intermediate variables could be considered as state variables. In the following each intermediate variable, except inputs, will be described as equations via other state variables

$$
\begin{aligned}
&y = W_4 \cdot x_4 \\
&x_4 = x_3 + W_8 \cdot x_2 \\
&x_3 = W_3 \cdot (x_2 + W_6 \cdot y) \\
&x_2 = W_2 \cdot (x_1 - W_7 \cdot x_4) \\
&x_1 = W_1 \cdot (s - W_5 \cdot y)
\end{aligned}
\quad .
$$

To express the output signal from the input signal the simultaneous equations are solved using replacement method.

$$x_2 = W_2 \cdot [W_1 \cdot (s - W_5 \cdot y) - W_7 \cdot x_4] = W_1 \cdot W_2 \cdot s - W_2 \cdot W_5 \cdot y - W_2 \cdot W_7 \cdot x_4$$

$$
\begin{aligned}
x_3 = W_3 \cdot [W_1 \cdot W_2 \cdot s - W_2 \cdot W_5 \cdot y - W_2 \cdot W_7 \cdot x_4 + W_6 \cdot y] = \\
= W_1 \cdot W_2 \cdot W_3 \cdot s - W_2 \cdot W_3 \cdot W_5 \cdot y - W_2 \cdot W_3 \cdot W_7 \cdot x_4 + W_3 \cdot W_6 \cdot y
\end{aligned}
$$

$$
\begin{aligned}
x_4 = W_1 \cdot W_2 \cdot W_3 \cdot s - W_2 \cdot W_3 \cdot W_5 \cdot y - W_2 \cdot W_3 \cdot W_7 \cdot x_4 + W_3 \cdot W_6 \cdot y + \\
+ W_8 \cdot [W_1 \cdot W_2 \cdot s - W_2 \cdot W_5 \cdot y - W_2 \cdot W_7 \cdot x_4] \Rightarrow
\end{aligned}
$$

$$x_4 = \frac{(W_1 \cdot W_2 \cdot W_3 + W_1 \cdot W_2 \cdot W_8) \cdot s - (W_1 \cdot W_2 \cdot W_3 \cdot W_5 - W_3 \cdot W_6 + W_1 \cdot W_2 \cdot W_5 \cdot W_8) \cdot y}{1 + W_2 \cdot W_3 \cdot W_7 + W_2 \cdot W_7 \cdot W_8}$$

$$y = \frac{W_4 \cdot (W_1 \cdot W_2 \cdot W_3 + W_1 \cdot W_2 \cdot W_8) \cdot s - (W_1 \cdot W_2 \cdot W_3 \cdot W_5 - W_3 \cdot W_6 + W_1 \cdot W_2 \cdot W_5 \cdot W_8) \cdot y}{1 + W_2 \cdot W_3 \cdot W_7 + W_2 \cdot W_7 \cdot W_8} \Rightarrow$$

$$W=\frac{W_1 \cdot W_2 \cdot W_3 \cdot W4 + W_1 \cdot W_2 \cdot W_4 \cdot W_8}{1+W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5 + W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_8 + W_2 \cdot W_3 \cdot W_7 + W_2 \cdot W_7 \cdot W_8 - W_3 \cdot W_4 \cdot W_6}$$

.

### 2.2.4.3. Determination of the transfer function by Mason's formula

For the transformation of the structures of the automatic control systems the **graph** theory, which is a branch of discrete mathematics while dealing with the topology of system elements, could be efficiently used. From this field stems the S.J. Mason's formula, the simplified form of which could be used for the simplification of structural diagrams that do not have hermit loops. As hermit loop are those feedback parts loops of structure diagram named, which does not have common elements.

**Simplified Manson's formula** is used by the structure diagram presented in the figure 2.13, where all feedback loops contain elements which are involved into some other feedback loop or where the hermit loops are absent or the following formula could be used for the simplification of structure diagrams: [2]

$$W=\frac{\Sigma W_{oj}}{1 \mp \Sigma W_{ki}}, \tag{2.11}$$

where $W_{oj}$ – transfer function of the j-th forward chain
$W_{ki}$ – transfer function of the i-th opened feedback loop.

If the feedback is negative, then $W_{ki}$ has a „+“-sign, in case of positive feedback the sign is „-“.

*Example 2.13.*

From the structure diagram represented in figure 2.13 two forward chains could be noticed:

$$W_{o1}=W_1 \cdot W_2 \cdot W_3 \cdot W_4$$
$$W_{o2}=W_1 \cdot W_2 \cdot W_4 \cdot W_8.$$

Also five feedback loops could be noticed here, the open loop transfer functions of which express as follows:

$$W_{k1}=W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5$$
$$W_{k2}=W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_8$$
$$W_{k3}=W_2 \cdot W_3 \cdot W_7$$
$$W_{k4}=W_2 \cdot W_7 \cdot W_8$$
$$W_{k3}=-W_3 \cdot W_4 \cdot W_6.$$

Replacing the found transfer functions into the formula (2.11) the result will be:

$$W=\frac{W_1 \cdot W_2 \cdot W_3 \cdot W4 + W_1 \cdot W_2 \cdot W_4 \cdot W_8}{1+W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5 + W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_8 + W_2 \cdot W_3 \cdot W_7 + W_2 \cdot W_7 \cdot W_8 - W_3 \cdot W_4 \cdot W_6}$$

.

**The Masons full formula** is used in cases when one has to deal with hermit loops. According to the definition of the hermit loops given above, in the figure 2.25 a corresponding structure of an automatic control system is presented. For simplification of this structure the formula [7] is used:

$$W = \frac{\Sigma W_{oj} \cdot \Delta_j}{\Delta} \, , \tag{2.12}$$

where $\Delta$ – determinant of the automatic control system
$\Delta_j$ – determinant of the j-th forward chain.

The determinant of the automatic control system is calculated as follows:

$$\Delta = 1 - \Sigma W_{ki} + \Sigma W_{ej} \cdot W_{ek}\big|_{j \neq k} - \Sigma W_{ej} \cdot W_{ek} \cdot W_{el}\big|_{j \neq k \neq l} + \ldots \, , \tag{2.13}$$

where $W_{ex}$ – transfer function of an open hermit loop.

The determinant of the forward chain is used similar to the system determinant to assign remaining untouched loops after deletion of the i-th forward chain.

*Example 2.14.*

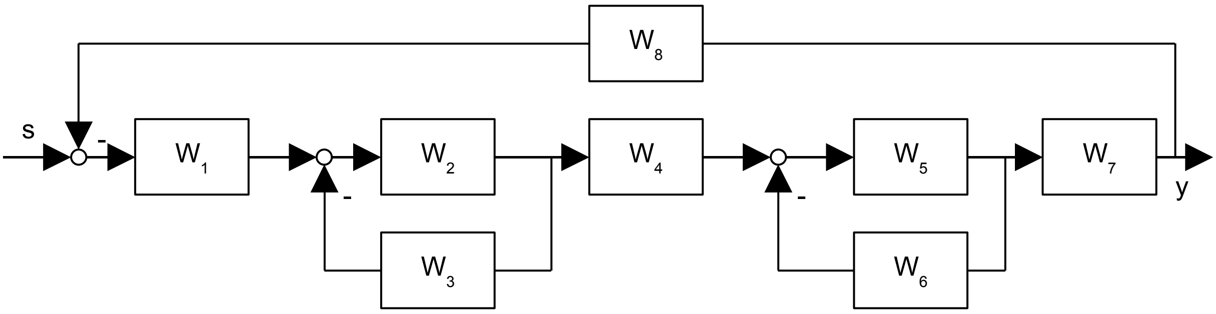



Figure 2.25. An automatic control system with hermit loops

The given structure diagram has three feedback loops, two of which don't have common elements or, one has to deal with two hermit loops.

From the structure diagram the transfer function of the forward chain

$$W_{o1} = W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_7$$

and three transfer functions of open feedback structures

$$W_{k1} = -W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_7 \cdot W_8$$
$$W_{k2} = -W_2 \cdot W_3 \qquad ,$$
$$W_{k3} = -W_5 \cdot W_6$$

two of which are hermit structures

$$W_{e1} = -W_2 \cdot W_3$$
$$W_{e2} = -W_5 \cdot W_6 \, .$$

To assign the determinant of the structure, the found transfer functions will be placed into equation (2.13)

$$\Delta = 1 + W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_7 \cdot W_8 + W_2 \cdot W_3 + W_5 \cdot W_6 + W_2 \cdot W_3 \cdot W_5 \cdot W_6 \, .$$

By deletion of the forward chain all loops will be broken, therefore its determinant equals to 1. Placing the results into equation (2.12) one gets as result the transfer function of the system:

$$W = \frac{W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_7 \cdot 1}{1 + W_1 \cdot W_2 \cdot W_4 \cdot W_5 \cdot W_7 \cdot W_8 + W_2 \cdot W_3 + W_5 \cdot W_6 + W_2 \cdot W_3 \cdot W_5 \cdot W_6} \, .$$

## 2.3. Description of the systems by state equations

In the division 1.2 state equations, which were presented mathematically with equation (1.7), were considered briefly.

$$\begin{aligned}\dot{\boldsymbol{x}}&=\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{u}\\ \boldsymbol{y}&=\boldsymbol{C}\cdot\boldsymbol{x}+\boldsymbol{D}\cdot\boldsymbol{u}\end{aligned}\ .$$

It is possible to compose the state equations for the continuous systems proceeding from the differential equations or structure diagrams

### 2.3.1. Determination of the state equations from the differential equations

*Example 2.15.*

Assigning state equation to the DC drive with controlled speed, familiar from the Example 2.8, the differential equation of the drive is written first

$$\begin{aligned}u_s(t)&=R\cdot i(t)+L\cdot\frac{d\,i(t)}{dt}+k_m\cdot\omega(t)\\ J\cdot\frac{d\,\omega(t)}{dt}&=k_m\cdot i(t)-T_k(t)\end{aligned}\ .$$

After arrangement the equations take the following form

$$\begin{aligned}\frac{d\,i(t)}{dt}&=-\frac{R}{L}\cdot i(t)-\frac{k_m}{L}\cdot\omega(t)+\frac{1}{L}\cdot u_s(t)\\ \frac{d\,\omega(t)}{dt}&=\frac{k_m}{J}\cdot i(t)-\frac{1}{J}\cdot T_k(t)\end{aligned}\ ,$$

which, if represented in matrix form, is

$$\underbrace{\begin{bmatrix}\frac{d\,i(t)}{dt}\\ \frac{d\,\omega(t)}{dt}\end{bmatrix}}_{\dot{\boldsymbol{x}}}=\underbrace{\begin{bmatrix}-\frac{R}{L} & -\frac{k_m}{L}\\ \frac{k_m}{J} & 0\end{bmatrix}}_{\boldsymbol{A}}\cdot\underbrace{\begin{bmatrix}i(t)\\ \omega(t)\end{bmatrix}}_{\boldsymbol{x}}+\underbrace{\begin{bmatrix}\frac{1}{L} & 0\\ 0 & -\frac{1}{J}\end{bmatrix}}_{\boldsymbol{B}}\cdot\underbrace{\begin{bmatrix}u_s(t)\\ T_k(t)\end{bmatrix}}_{\boldsymbol{u}}\ .$$

By a drive with speed control usually both as current as well speed are monitored. The reasons for the monitoring of the current are more detailed considered by the synthesis of automatic control systems. If to proceed from the state equation and output variables the output equation could be expressed as follows:

$$\underbrace{\begin{bmatrix}i(t)\\ \omega(t)\end{bmatrix}}_{\boldsymbol{y}}=\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\boldsymbol{C}}\cdot\underbrace{\begin{bmatrix}i(t)\\ \omega(t)\end{bmatrix}}_{\boldsymbol{x}}+\underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}_{\boldsymbol{D}}\cdot\underbrace{\begin{bmatrix}u_s(t)\\ T_k(t)\end{bmatrix}}_{\boldsymbol{u}}\ .$$

*Example 2.16.*

Applying position control to the drive described in the previous example, the differential equation will present itself in following

$$\frac{d\,i(t)}{dt} = -\frac{R}{L}\cdot i(t) - \frac{k_m}{L}\cdot\omega(t) + \frac{1}{L}\cdot u_s(t)$$
$$\frac{d\,\omega(t)}{dt} = \frac{k_m}{J}\cdot i(t) - \frac{1}{J}\cdot T_k(t)$$
$$\frac{d\,\varphi(t)}{dt} = \omega(t)$$

and the state equations will take the following form

$$\underbrace{\begin{bmatrix} \frac{d\,i(t)}{dt} \\ \frac{d\,\omega(t)}{dt} \\ \frac{d\,\varphi(t)}{dt} \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{k_m}{L} & 0 \\ \frac{k_m}{J} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} i(t) \\ \omega(t) \\ \varphi(t) \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{J} \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} u_s(t) \\ T_k(t) \end{bmatrix}}_{\boldsymbol{u}}$$

$$\underbrace{\begin{bmatrix} i(t) \\ \omega(t) \\ \varphi(t) \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{C}} \cdot \underbrace{\begin{bmatrix} i(t) \\ \omega(t) \\ \varphi(t) \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{D}} \cdot \underbrace{\begin{bmatrix} u_s(t) \\ T_k(t) \end{bmatrix}}_{\boldsymbol{u}}.$$

### 2.3.2. Determination of the state equations from the structure diagrams

*Example 2.17.*

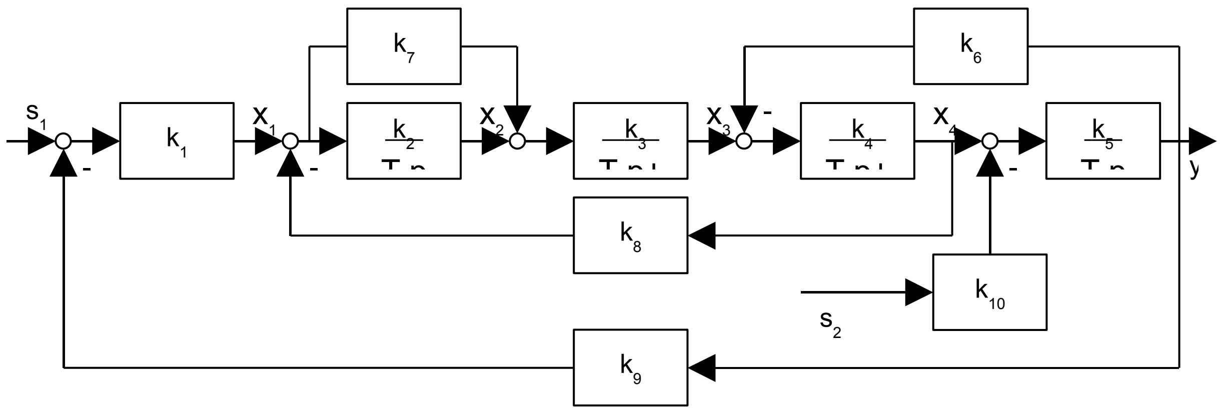



Figure 2.26. An automatic control system.

Using the principle of an structure diagram with open loops, one may written

$$y = \frac{k_5}{T_5\cdot p}\cdot\left(x_4 - k_{10}\cdot s_2\right)$$

$$x_4 = \frac{k_4}{T_4\cdot p+1}\cdot\left(x_3 - k_6\cdot y\right)$$

$$x_3 = \frac{k_3}{T_3\cdot p+1}\cdot\left[x_2 + k_7\cdot\left(x_1 - k_8\cdot x_4\right)\right]$$

$$x_2 = \frac{k_2}{T_2 \cdot p} \cdot (x_1 - k_8 \cdot x_4)$$

$$x_1 = k_1 \cdot (s_1 - k_9 \cdot y) \ .$$

By the arrangement of the equations the operator variable will be transferred to the left side of the equality sign and the state variables ordered by their indices. So as in the last equation the operator variable is missing, it will placed in other equations

$$p\,y = \frac{k_5}{T_5} \cdot x_4 - \frac{k_5 \cdot k_{10}}{T_5} \cdot s_2$$

$$p\,x_4 = -\frac{k_4 \cdot k_6}{T_4} \cdot y - \frac{1}{T_4} \cdot x_4 + \frac{k_4}{T_4} \cdot x_3$$

$$p\,x_3 = -\frac{k_1 \cdot k_3 \cdot k_7 \cdot k_9}{T_3} \cdot y - \frac{k_3 \cdot k_7 \cdot k_8}{T_3} \cdot x_4 - \frac{1}{T_3} \cdot x_3 + \frac{k_3}{T_3} \cdot x_2 + \frac{k_1 \cdot k_3 \cdot k_7}{T_3} \cdot s_1$$

$$p\,x_2 = -\frac{k_1 \cdot k_2 \cdot k_9}{T_2} \cdot y - \frac{k_2 \cdot k_8}{T_2} \cdot x_4 + \frac{k_1 \cdot k_2}{T_2} \cdot s_1 \ .$$

Representing the matrixes in general form

$$\underbrace{\begin{bmatrix} p\,y \\ p\,x_4 \\ p\,x_3 \\ p\,x_2 \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} y \\ x_4 \\ x_3 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}}_{\boldsymbol{u}}$$

$$\underbrace{[y]}_{\boldsymbol{y}} = \underbrace{[c_{11} \quad c_{12} \quad c_{13} \quad c_{14}]}_{\boldsymbol{C}} \cdot \underbrace{\begin{bmatrix} y \\ x_4 \\ x_3 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{[d_{11} \quad d_{12}]}_{\boldsymbol{D}} \cdot \underbrace{\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}}_{\boldsymbol{u}}$$

and applying the comparison principle of constants it could be expressed as follows

$$a_{12} = \frac{k_5}{T_5} \qquad a_{21} = -\frac{k_4 \cdot k_6}{T_4} \qquad a_{22} = -\frac{1}{T_4} \qquad a_{23} = \frac{k_4}{T_4} \qquad a_{31} = -\frac{k_1 \cdot k_3 \cdot k_7 \cdot k_9}{T_3}$$

$$a_{32} = -\frac{k_3 \cdot k_7 \cdot k_8}{T_3} \qquad a_{33} = -\frac{1}{T_3} \qquad a_{34} = \frac{k_3}{T_3} \qquad a_{41} = -\frac{k_1 \cdot k_2 \cdot k_9}{T_2} \qquad a_{42} = -\frac{k_2 \cdot k_8}{T_2}$$

$$b_{12} = -\frac{k_5 \cdot k_{10}}{T_5} \qquad b_{31} = \frac{k_1 \cdot k_3 \cdot k_7}{T_3} \qquad b_{41} = \frac{k_1 \cdot k_2}{T_2} \qquad c_{11} = 1$$

The unrevealed constants equal to zero.

## 2.4. Description of systems by characteristic curves

If it is impossible to use a mathematical approach for the description of the system, one has to deal with an improving device with incomplete data for it, for instance, or with a black box, the characteristic curves of the device or process will be studied- A characteristic curve is the graphical description of the process. Two kinds of characteristic curves are distinguished – dynamic and static. Dynamic characteristic curve characterises the transient processes or transition of the output variable of the device from one value to another, and this in the time domain. Those characteristics are also called transient characteristics. Static characteristics describe the dependence of the system output on the all variables, the disturbances included, by the condition that transient processes in the device are terminated. Which of characteristic to use for the study of the device depends on the objective of the study. The dynamic characteristic describes fast processes and they are used if the quality and the duration of the transient process are in importance, by an automatic miller, for instance. The static characteristics are in interest if the parameters of the transient process do not influence essentially the operation of the device, usual water pump, for example.

### 2.4.1. Dynamic characteristic curve

Dynamic characteristic curve represents the response of the device or of the process to the input signal. It could be named as signal echo. So as the input signals could be different, (different amplitude, different time progress of the signal, etc) then for the comparison of different devices amongst, the following input functions are used: [3]:

unit jump
unit impulse;
linear function;
cosine function.

The time development of **unit jump** describes the characteristic curve in the figure 2.27. Mathematical representation of this function is the following

$$\sigma(t)=\begin{matrix} 0 & , kui\, t<0 \\ 1 & , kui\, t\geq 0 \end{matrix} . \qquad (2.14)$$

In case if one has to deal with a real device, the mathematical description of what is sought, then a signal will be applied to the input of the device, corresponding to the unit jump, and output signal will be recorded. After the termination of transient processes the mathematical presentation of the device will be assessed by the recorded curve, from the figure 2.28, for example, could be read

$$h(t)=K\cdot(1-e^{-\frac{t}{T}})$$

If one has to deal with engineering design of a device which is wanted to simulate, then the differential equation describing the device will be solved by the condition, where the input signal is permanently 1 and time will be let run to infinity.

Joonis 2.27. Ühikhüpe

Joonis 2.28. Hüppekaja

Joonis 2.29. Ühikimpulss

The progress of a **unit impulse** in time is described with the characteristic curve in figure 2.29. Mathematical description of this function is the following:

$$\delta(t)=\begin{matrix} 0 & ,kuit\neq 0 \\ \infty & ,kuit=0 \end{matrix} \,. \qquad (2.15)$$

Also is this function called Dirac's function. So as the infinity is an indefinite value, then in practice the impulse is realised so that its area equals to 1 whereas its duration approaches to zero. Thereby the amplitude must not exceed the permissible for the device, maximum signal, for instance, if the isolation of the connection cord is intended for the use up to 1 kV, then there must not be applied the voltage of higher amplitude. As for the jump echo it is possible for the impulse echo determine the characteristic equation, for example, from the figure 2.30.

$$w(t)=\frac{K}{T}\cdot e^{-\frac{t}{T}} \,.$$

As in the figure 2.28 as well in the figure 2.30 the same device is described, to which different input signals were given. If one has to do with the same system, the should there exist a mathematical relation between those two signals, which is

$$h(t)=\int w(t)dt \qquad (2.16)$$

and these relations are graphically presented in the figure 2.31.

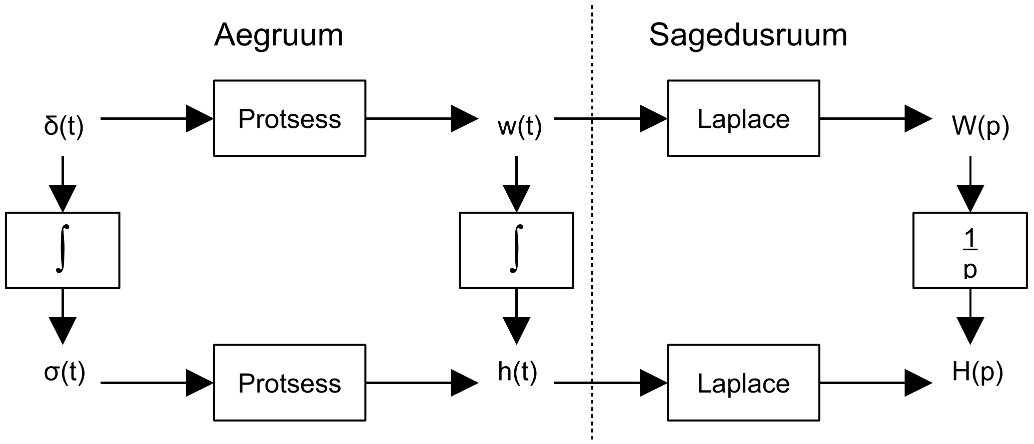



Figure 2.31. Graphical relation of the unit impulse and of the jump

The progress of the **linear function** in time is described in the figure 2.32. Mathematical presentation of this function is

$$R(t)=K_1\cdot t \,. \qquad (2.17)$$

The alteration of the output of a device produced by linear function is indicated by f (t), for instance, the expression of the echo of a linear function: is

$f(t) = K \cdot (t - T + T \cdot e^{-\frac{t}{T}})$

In the fig8ure 2.33 the signal echo of the system investigated with two previous functions is presented, from which it is possible to determine the time constant of the system. If to extend the tangent to the ordinate axis, then through the intersection point it is possible to determine the gain of the device.

Joonis 2.32. Lineaarfunktsioon

Joonis 2.33. Signaalikaja

The figure 2.34 describes the progress of the **cosine function** in time**.** Mathematical description of this function is

$$s(t) = \hat{s} \cdot \cos(\omega \cdot t)\ . \qquad (2.18)$$

Joonis 2.34. Koosinusfunktsioon

Joonis 2.35. Signaalikaja

Te ... of time
Th ... the app
int ... d the pha
the ... evice equ
sig ... tion of th
fig

## 2.4.2. Static characteristic

Static characteristic describes the system in the evolved state that could be expressed from the operator form of an automatic control system presented in the figure 2.36. [3], [8]

$$y(p)=\frac{W_R \cdot W_P}{1 \mp W_R \cdot W_P} \cdot s(p)+\frac{W_P}{1 \mp W_R \cdot W_P} \cdot n(p)\,. \qquad (2.19)$$

Joonis 2.36. Automaatjuhtimissüsteem

The equation (2.19) describes the output value of a closed system, however, for an open system the equation will have another form

$$y(p)=W_R \cdot W_P \cdot s(p)+W_P \cdot n(p)\,. \qquad (2.20)$$

To determine the evolved value of the output variable the given above equation will be solved (depending on the system) as follows

$$y_{st}(t)=\lim_{p \to 0} p \cdot y(p) \,, \qquad (2.21)$$

which will ... 'or the real existing sys ... if possible, modify the ... follows, the result could

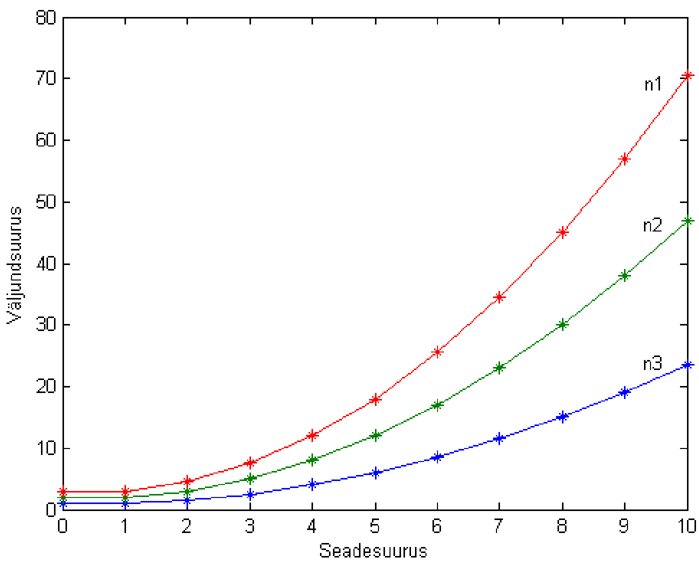



Figure 2.37. Static characteristic

From this graph it is easy to read the value of the output variable by a given set value and given disturbance value. Also it gives estimation about the change of the output value by changing of the disturbance. Formerly, the linearization of those graphs around the operation point was used, now these characteristics are used less. In the use are remained some concepts from the static characteristics

**Static** or **evolved output value** is the value that the output variable takes if from the beginning of the transient process has passed infinite time. It could be calculated by the equation (2.21).

The **permanent control error** is the difference between the desired and realized values of the output variable. Mathematically it could be expressed as

$$x_{d\ st} = \lim_{p \to 0} \left( K \cdot s(p) - y(p) \right) \ , \qquad (2.22)$$

where K – planned gain of the system.

The **efficiency of the regulator** in holding the system in a stable operation describes the ability of the regulator to reduce the influence of disturbances on the operation of the system. The evaluation of the efficiency could be expressed from the equations (2.19) and (2.20), where the open system descries the system without regulator ( $W_R$ = 1; s(p) = 0) and the closed system describes the system with regulator (s(p) = 0). Their ratio is the rate of the efficiency

$$\frac{y_{reg}}{y} = \frac{1}{1 \mp W_R \cdot W_P} \ . \qquad (2.23)$$

From the equation (2.23) might be concluded, that in the system with closed loop the influence of the disturbance will be decreased by 1-$W_R W_P$ times

## 2.5. Standard links

In the division 2.2.2 it was concluded, that if to control voltage of a RCL circuit and the speed of an DC motor by the input voltage, then both systems behave similarly an it is possible to describe them by similar mathematical equations. Thus, if there exist mathematically similar control objects, then. consequently there exist mathematically similar control devices, from where it follows in turn, that if for one control object matches one control device, then for another similar object could match a control device which is similar to the control device of the first control object. In other words - one has to do with standard solutions and the used links of the automatic control systems are therefore named standard links.

### 2.5.1. Proportional link

Proportional link is also named an amplification link or a link without inertia, or for short P/link. The output signal of o P/link changes simultaneously with input jump signal and also jump-wise and without delay. The difference of the signal levels is due to the gain only

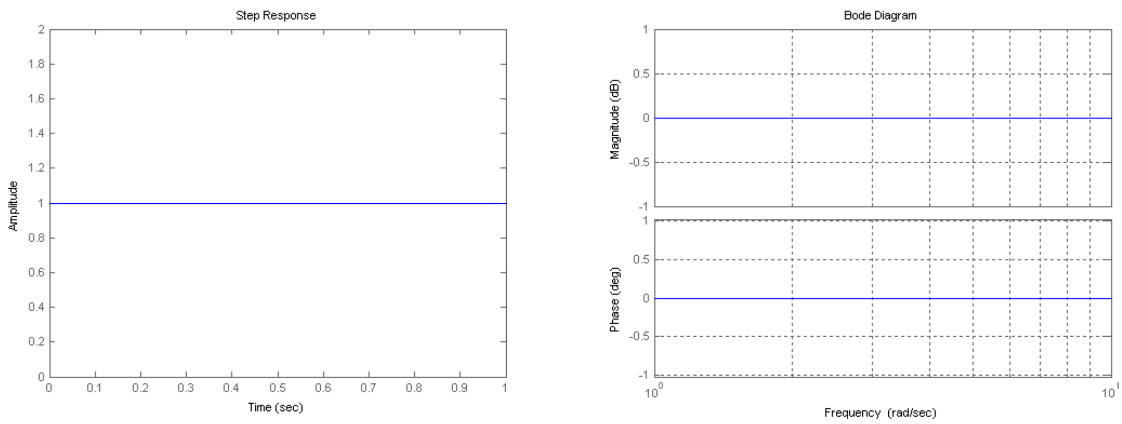
Joonis 2.38. P-lüli

The differential equation: $y(t) = K \cdot s(t)$ (2.24)

The transfer function $W(p) = K$ (2.25)

The impulse echo $w(t) = K \cdot \delta(t)$ (2.26)

The jump echo: $h(t) = K \cdot \sigma(t)$ (2.27)




a) b)

Figure 2.39. A P-link K=1. a) The jump echo, b) the Bode diagram

### 2.5.2. Integration link

The integrating link is also called an astatic link and I-link. The output signal of an ideal integrating link increases (or decreases) continuously with a constant speed if $x_s \neq 0$ and is constant. The speed is determined by the value of the jump in the input. By a real integrating link (described by $IT_1$-loink) growth ratio

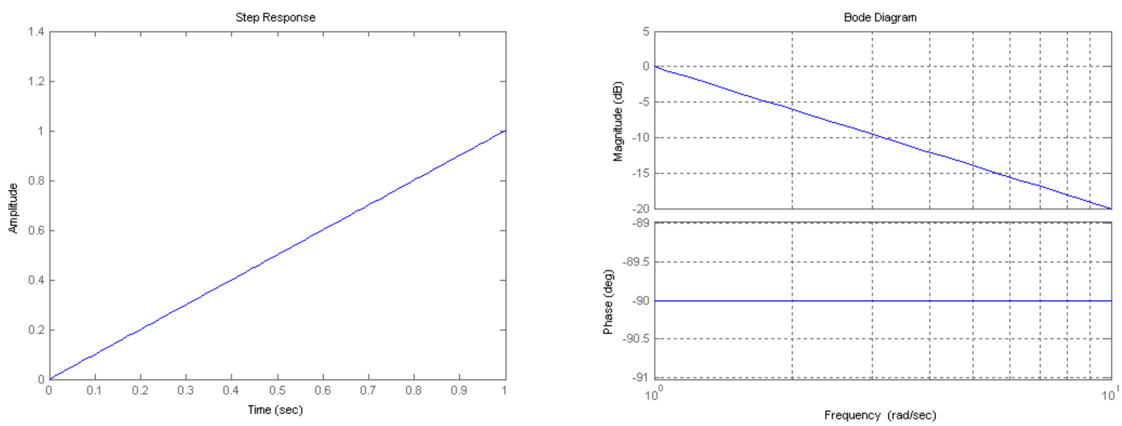
of the output signal is zero in first moment and grows slowly to the final speed.

The differential equation $\dot{y}(t)=K\cdot u(t)$ (2.28)

The transfer function: $W(p)=\frac{K}{p}$ (2.29)

The impulse echo: $w(t)=K\cdot\sigma(t)$ (2.30)

The jump echo $h(t)=K\cdot t$ (2.31)




a) b)

Figure 2.41. An I-link, K=1. a) The jump echo, b) the Bode diagram

### 2.5.3. Differentiation link

The another name of the differentiation link is D-link. The output signal of an ideal differentiation link is a hyper-short impulse of infinite great amplitude. For a real differentiation link (described as $DT_1$-link) The output signal grows very rapidly to a final value and diminishes thereafter gradually with reducing speed to zero

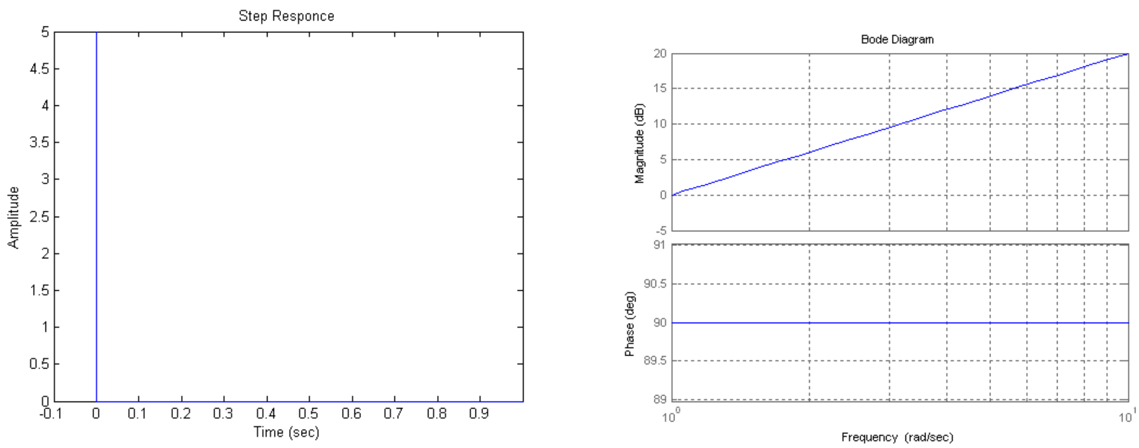
The differential equation $y(t)=K\cdot\dot{s}(t)$ (2.32)

The transfer function: $W(p)=K\cdot p$ (2.33)

The impulse echo $w(t)=K\cdot\frac{d}{dt}\delta(t)$ (2.34)

The jump echo

: $h(t)=K\cdot\delta(t)$ (2.35)




a) b)

Figure 2.43. D-link K=1. a) Jump echo, b) Bode diagram

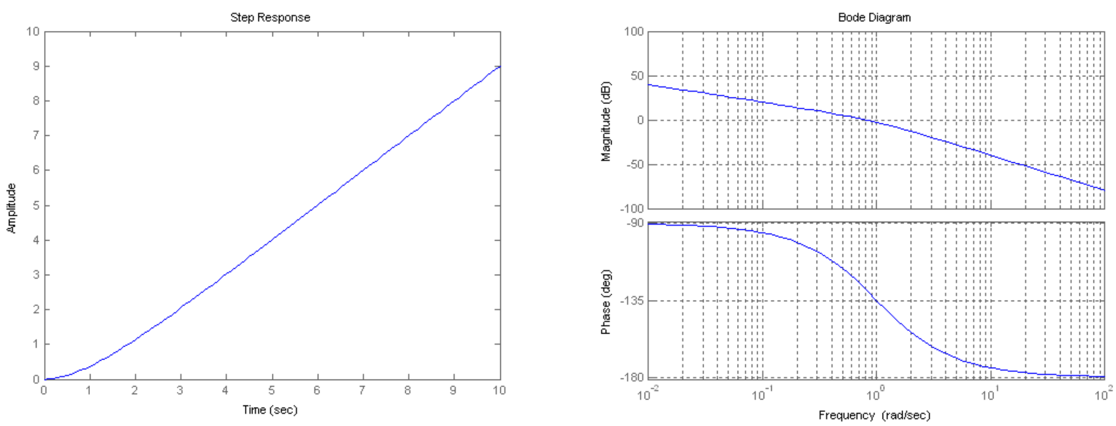
### 2.5.4. Integration link with time constant

Integration link with a time constant or short $IT_1$ describes a real integration link, which, different from the ideal one das distortions in the output.

Differential equation $T\cdot\ddot{y}(t)+\dot{y}(t)=K\cdot s(t)$

Joonis 2.44. $IT_1$-lüli

Transfer function: $W(p)=\frac{K}{p\cdot(1+T\cdot p)}$ (2.37)




a) b)

Figure 2.45. $IT_1$-llinkK=1, T=1 s. a) jump echo, b) Bode diagram

### 2.5.5. Differentiation link with time constant

Differentiation link with a time constant or short $DT_1$ describes a real differentiation link that, different from the ideal one has distortions in its output

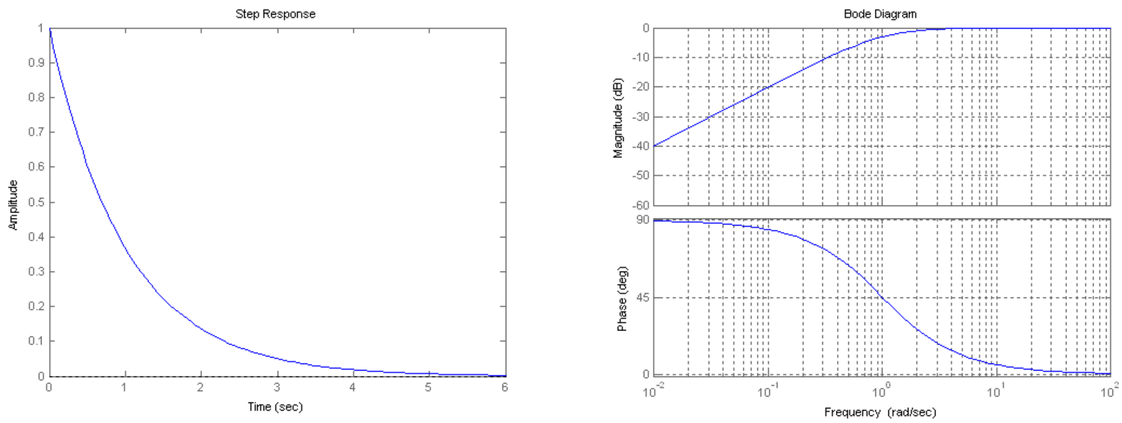
Joonis 2.46. $DT_1$-lüli

Differential equation: $T\cdot\dot{y}(t)+y(t)=K\cdot\dot{s}(t)$ (2.40)

Transfer function: $W(p)=\dfrac{K\cdot p}{1+T\cdot p}$ (2.41)

Impulse echo: $w(t)=\dfrac{K}{T}\delta(t)-\dfrac{K}{T^2}\cdot e^{-\frac{t}{T}}$ (2.42)

Jump echo: $h(t)=\dfrac{K}{T}\cdot e^{-\frac{t}{T}}$ (2.43)




a) b)

Figure 2.47. $DT_1$-link K=1, T=1 s. a) jump echo, b) Bode diagram

### 2.5.6. Delay link

Delay link behaves like P-link, but reacts to the input with certain delay. The delay link is noticed as $PT_h$-link.

*Notion!* Bode diagram is built with MathCAD software

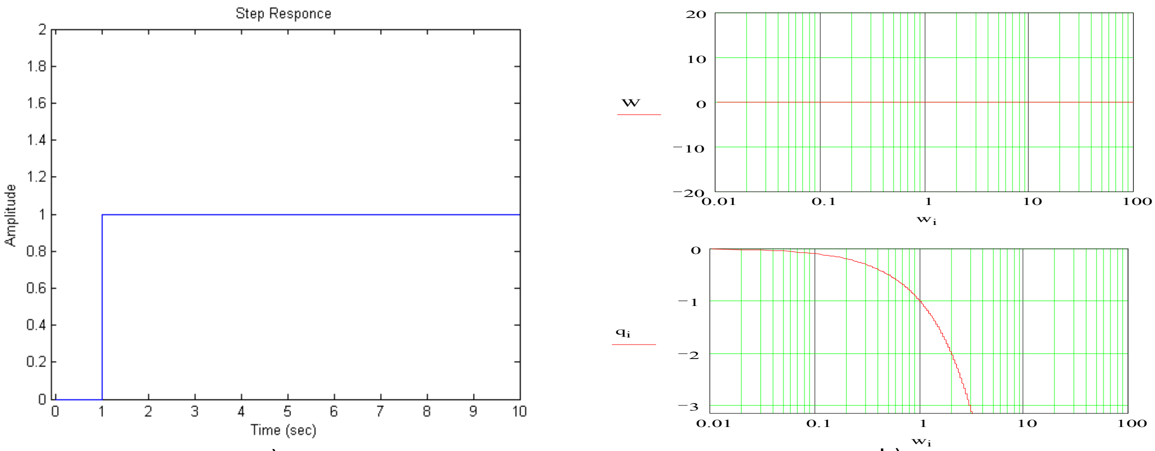
Joonis 2.48. $PT_h$-lüli

Differential equation $y(t)=K\cdot s(t-T_h)$ (2.44)

Transfer function: $W(p)=K\cdot e^{-T_h\cdot p}$ (2.45)

Impulse echo $w(t)=K\cdot\delta(t-T_h)$ (2.46)

Jump echo: $h(t)=K\cdot\sigma(t-T_h)$ (2.47)




a) b)

Figure 2.49. $PT_h$-link K=1, $T_h$=1 s. a) jump echo, b) Bode diagram

## 2.5.7. Aperiodic link

Aperiodic link is also called inertial link, relaxation link and $PT_1$-link. The output signal starts to change immediately, first with maximal speed, thereafter with gradually reducing speed until it reaches maximal value after (3...5)T. The transient characteristic represents an exponent curve

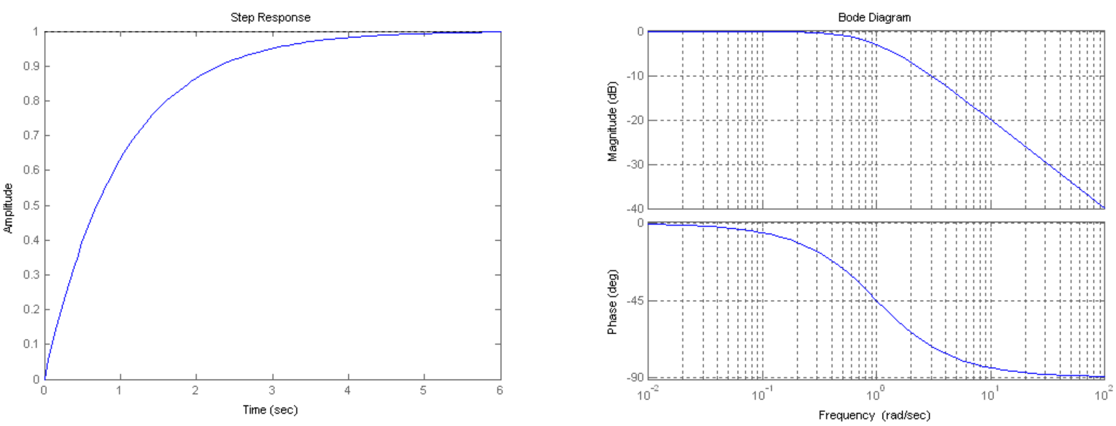
Joonis 2.50. $PT_1$-lüli

Differential equation $T\cdot\dot{y}(t)+y(t)=K\cdot s(t)$ (2.48)

Transfer function $W(p)=\frac{K}{1+T\cdot p}$ (2.49)

Impulse echo: $w(t)=K\cdot e^{-\frac{t}{T}}$ (2.50)

Jump echo $h(t)=K\cdot(1-e^{-\frac{t}{T}})$ (2.51)




a) b)

Figure 2.51. $PT_1$-linki K=1, T=1 s. a) jump echo, b) Bode diagram

### 2.5.8. Oscillator link

The oscillator link could be named as $PT_2$-link, therefore, that it has two time constants already. For better understanding of the operation of the oscillator link one has to introduce some new quantities. From the examples 2.12 and 2.13 the following transfer function is known

$$W(p)=\frac{K}{1+T_2 p+T_1 T_2 p^2} , \qquad (2.52)$$

From which, while substituting

$$\omega_0=\frac{1}{\sqrt{T_1 T_2}} , \qquad (2.53)$$

where $\omega_0$ – the characteristic of the object or radian frequency of un-damped oscillation

$$\alpha=\frac{1}{2T_1} , \qquad (2.54)$$

where α – damping factor

it is possible to express the equation (2.52) in other form

$$W(p)=\frac{K\cdot\omega_0^2}{\omega_0^2+2\alpha\cdot p+p^2} . \qquad (2.55)$$

Also it is possible to express with the damping factor and radian frequency of characteristic other parameters characterising the system

$$\omega_s=\sqrt{\omega_0^2-\alpha^2} , \qquad (2.56)$$

where ωs – radian frequency of damping oscillation

$$T=\frac{2\pi}{\omega_s} , \qquad (2.57)$$

where T – period of damping oscillations

$$\chi=\frac{\alpha}{\omega_0} , \qquad (2.58)$$

where χ – dan

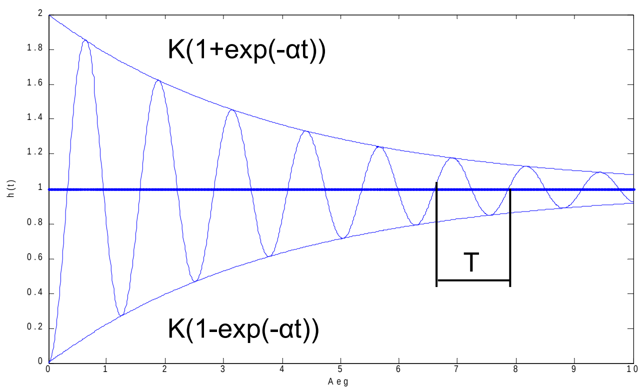


Figure 2.52. An example of the transient characteristic of an oscillation link

Depending on the damping factor three kinds of oscillation links are distinguished

*Damping factor $\chi > 1$*

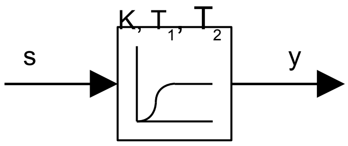


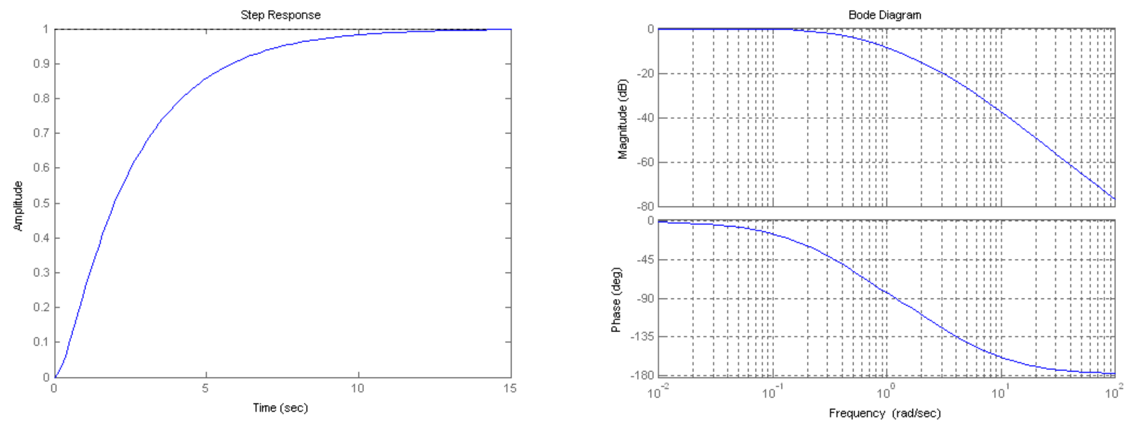

Figure 2.53. $PT_1$-link

Differential equation $$T_1 \cdot T_2 \cdot \ddot{y}(t) + (T_1 + T_2) \cdot \dot{y}(t) + y(t) = K \cdot s(t) \quad (2.59)$$

Transfer function: $$W(p) = \frac{K}{(1 + T_1 \cdot p) \cdot (1 + T_2 \cdot p)} \quad (2.60)$$

Impulse echo: $$w(t) = \frac{K}{T_1 - T_2} \cdot \left(e^{-\frac{t}{T_1}} - e^{-\frac{t}{T_2}}\right) \quad (2.61)$$

Jump echo $$h(t) = K \cdot \left[1 - \frac{1}{T_1 - T_2} \cdot \left(T_1 \cdot e^{-\frac{t}{T_1}} - T_2 \cdot e^{-\frac{t}{T_2}}\right)\right] \quad (2.62)$$




a) b)

Figure 2.54. $PT_1$-link K=1, $T_1$=0,3 s, $T_2$= 2,4 s. a) jump echo, b) Bode diagram

*Damping factor χ = 1*

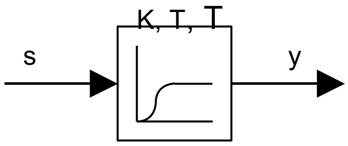


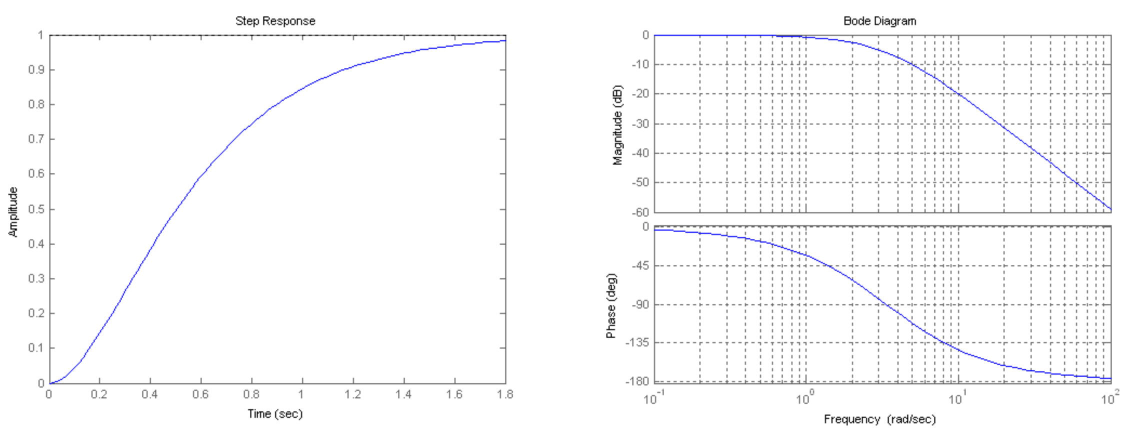

Figure 2.55. $PT_1$-link

Differential equation $\quad T^2 \cdot \ddot{y}(t) + 2 \cdot T \cdot \dot{y}(t) + y(t) = K \cdot s(t)$ (2.63)

Transfer function: $\quad W(p) = \dfrac{K}{(1 + T \cdot p)^2}$ (2.64)

Impulse echo: $\quad w(t) = \dfrac{K}{T^2} \cdot t \cdot e^{-\frac{t}{T}}$ (2.65)

Jump echo: $\quad h(t) = K \cdot [1 - (1 + \dfrac{t}{T}) \cdot e^{-\frac{t}{T}}]$ (2.66)




a) b)

Figure 2.56. $PT_1$-link K=1, $T_1$=0,3 s, $T_2$= 0,3 s. a) jump eco, b) Bode diagram

*Damping factor 0 < χ < 1*

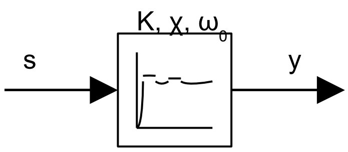


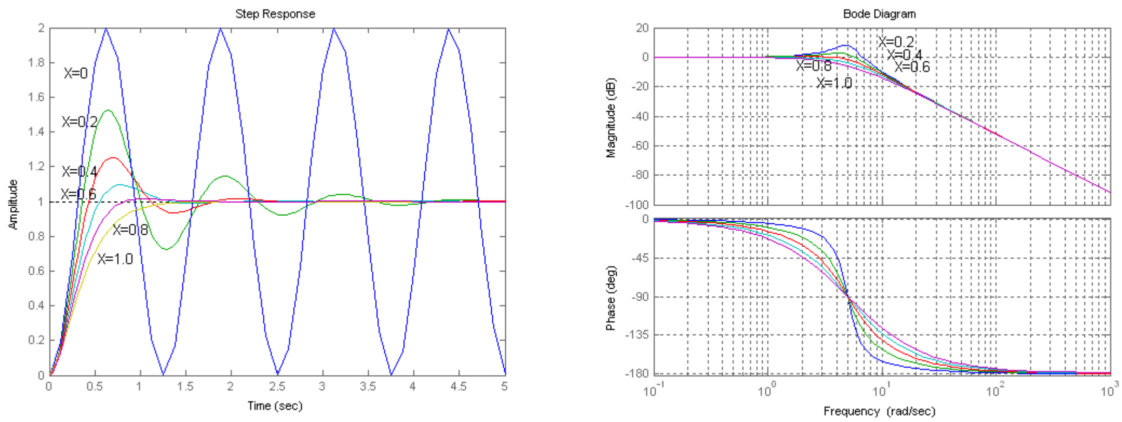

Figure 2.57. $PT_1$-link

Differential equation $\quad \dfrac{1}{\omega_0^2} \cdot \ddot{y}(t) + \dfrac{2 \cdot \chi}{\omega_0} \cdot \dot{y}(t) + y(t) = K \cdot s(t)$ (2.67)

Transfer function: $$W(p)=\frac{K}{\frac{1}{\omega_0^2}\cdot p^2+\frac{2\cdot\chi}{\omega_0}\cdot p+1} \tag{2.68}$$

Impulse echo $$w(t)=K\cdot\frac{\omega_0}{\sqrt{1-\chi^2}}\cdot e^{-\chi\cdot\omega_0\cdot t}\cdot\sin(\sqrt{1-\chi^2}\cdot\omega_0\cdot t) \tag{2.69}$$

Jump echo
$$h(t)=K\cdot\{1-e^{-\chi\cdot\omega_0\cdot t}\cdot[\cos(\sqrt{1-\chi^2}\cdot\omega_0\cdot t)+\frac{\chi}{\sqrt{1-\chi^2}}\cdot\sin(\sqrt{1-\chi^2}\cdot\omega_0\cdot t)]\} \tag{2.70}$$




a) b)

Figure 2.58. $PT_1$-link K=1, $\chi$=0...1, $\omega_0$=5 $s^{-1}$. a) Jump echo, b) Bode diagram

## 2.5.9. Proportional-integration link

PI-link combines proportional and integration links.
This link is used in regulators mainly

Differential equation: $$\dot{y}(t)=K_R\cdot(\frac{1}{T_I}\cdot s(t)+\dot{s}(t))$$

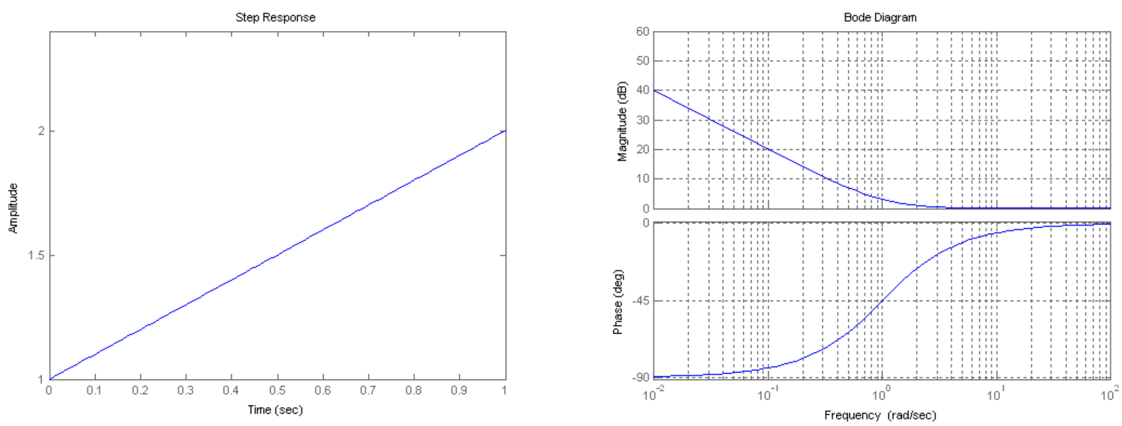
Joonis 2.59. PI-lüli

Transfer function: $$W(p)=K_R\cdot\frac{1+T_I\cdot p}{T_I\cdot p} \tag{2.72}$$

Impulse echo $$w(t)=K_R\cdot(\delta(t)+\frac{1}{T_I}\cdot\sigma(t)) \tag{2.73}$$

Jump echo $$h(t)=K_R\cdot(1+\frac{1}{T_I}\cdot t) \tag{2.74}$$

For the connection of the parameters of a PI-link with gain of an I-link the following




a) b)

Figure 2.60. PI-link K=1, $T_I$=1 s. a) jump echo, b) Bode diagram

## 2.5.10. Proportional-differentiation link

PD-link combines proportional link and differentiation link. Given link is used in regulators mainly

Differential equation: $y(t)=K_R\cdot(s(t)+T_D\cdot\dot{s}(t))$

Transfer function $W(p)=K_R\cdot(1+T_D\cdot p)$

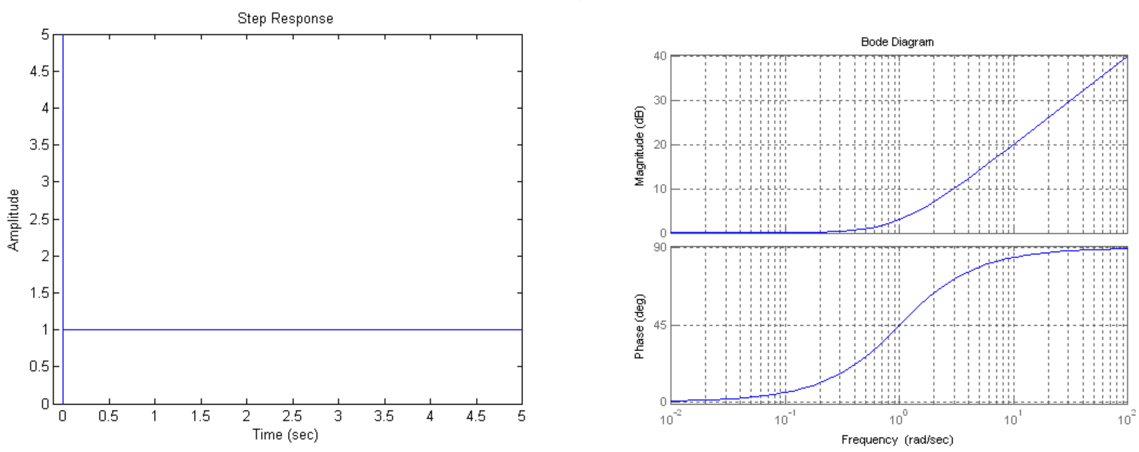
Joonis 2.59. PI-lüli

Impulse echo: $$w(t)=K_R\cdot(\delta(t)+T_D\cdot\frac{d}{dt}\cdot\delta(t)) \qquad (2.78)$$

Jump echo: $$h(t)=K_R\cdot(\sigma(t)+T_D\cdot\delta(t)) \qquad (2.79)$$

To connect the parameters of a PD-link with the gain of a D-link the following relation holds:

$$K_I=K_R\cdot T_D\ . \qquad (2.80)$$




a) b)

Figure 2.62. PD-link K=1, $T_D$=1 s. a) jump echo, b) Bode diagram

## 2.5.11. Proportional-integration- differentiation link

PID-link combines all three main elements – P-link, I-link and D-link. PID-regulators are most widespread control devices, whereas PI- and PD-regulators are realised with PID-regulator, where the gain of the D- or I link equals zero.

Joonis 2.63. PID-lüli

Differential equation $\dot{y}(t)=K_R\cdot(\frac{1}{T_I}\cdot s(t)+\dot{s}(t)+T_D\cdot\ddot{s}(t))$ (2.81)

Transfer function: $W(p)=K_R\cdot\frac{1+T_I\cdot p+T_I\cdot T_D\cdot p^2}{T_I\cdot p}$ (2.82)

Impulse echo $w(t)=K_R\cdot(\delta(t)+\frac{1}{T_I}\cdot\sigma(t)+T_D\cdot\frac{d}{dt}\cdot\delta(t))$ (2.83)

Jump echo: $h(t)=K_R\cdot(\sigma(t)+\frac{1}{T_I}\cdot t+T_D\cdot\delta(t))$ (2.84)

Joonis 2.64. PID-lüli K=1, $T_I$= 50 s, $T_D$=0.5 s. a) hüppekaja, b) Bode diagramm

Express the dependence of the output voltage of the system presented in figure 2.65 on the input voltage with a differential equation. Transform the found equation into operator form and determine the transfer function of the system. Repeat the same by the condition that output variable is the current.

Joonis 2.65. RC-ahel

*Exercise 2.*

Determine the transfer function of the system represented in figure 2.66.

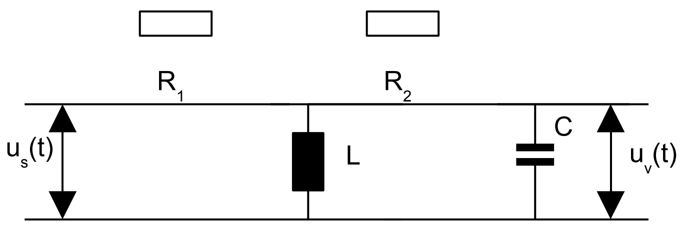



Figure 2.66. RLC-circuit

*Exercise 3.*

Find th ... function of ... ture diagra ... nted in figu

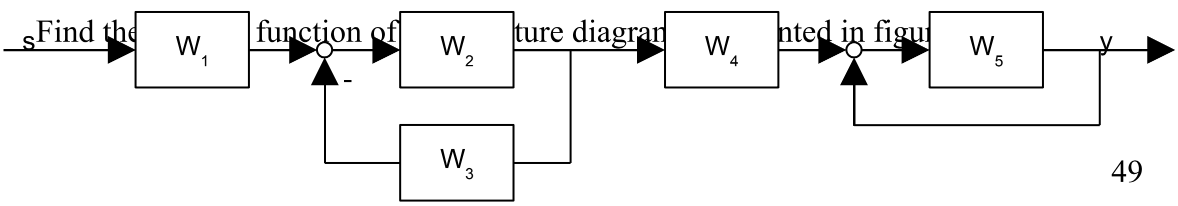


Figure 2.67.Structure diagram of an automatic control system

*Exercise 4.*

Find the transfer function of the structure diagram represented in figure 2.68

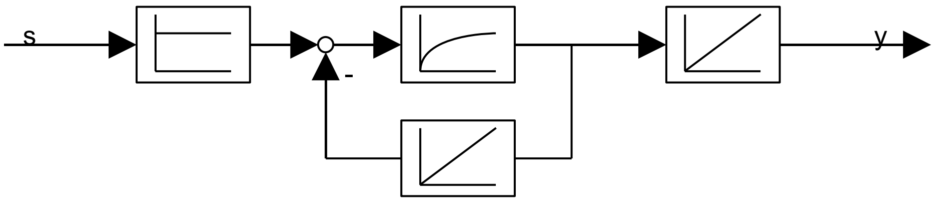



Figure 2.68.Structure diagram of an automatic control system

# 3. ANALYSIS OF THE AUTOMATIC CONTROL SYSTEMS

It is essential by the analysis of automatic control systems to determine the main properties of the system. These are the controllability and observability, but also stability and the quality of dynamics. A system is **controllable**, if during a finite time interval $t_a \leq t \leq t_l$ exist such a control action u (t), that takes the system from given initial state into the final state or sufficiently close to it. The system is **fully controllable,** if there exist a control action with which is would be possible to bring the mapping point of a system to the zero point of the coordinates. The system is

**partly controllable** if the control action does not influence all of state variables or if a part of state variables does not affect the system output.

R. Kalman has formulated the **criteria of controllability** depending on the of the order of the differential equation of the controllable part of the system. The system is fully controllable, if the order of the differential equation of the controllable part of the system equals to the order of its state matrix (matrix **A**). The system is **partly controllable**, if the order of the differential equation is less than the order of the state equation, and the system is **not controllable** if the order of the differential equation is zero.

The observability of the system characterises the dependence between is outputs and state variables, whereby the system could be or fully or partly observable. By Kalman, a system is **observable,** if the order of the differential equation of its observable part equals to the order of the state equation. System is **observable**, if the order of its differential equation is less of the order of its state equation, and **not observable** if the order of its differential equation is zero

The theory, considering the **stability** of the system is based on the investigation of the stability problems of the solutions of differential equations, known from mathematics. By the classical methods of automatic control, the system stability is controlled by the transfer function and characteristic equation.

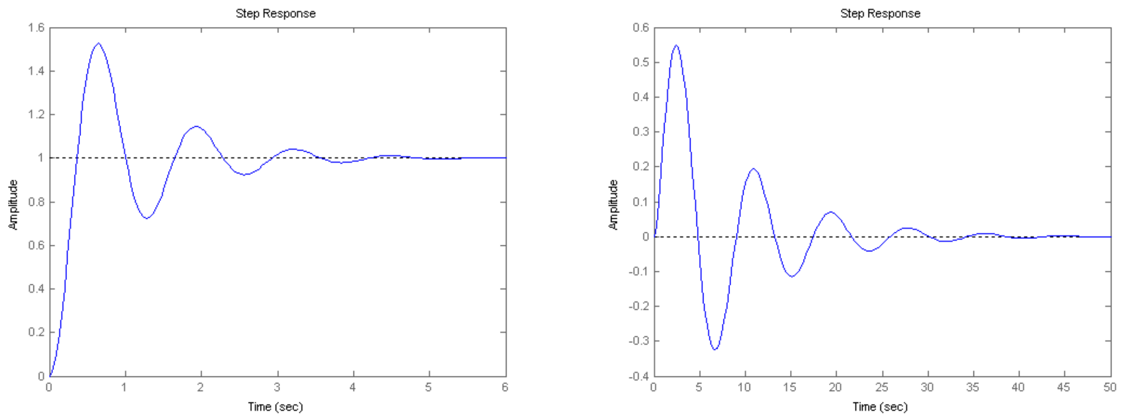
As the **static accuracy** of the system the correlation of the system, output to the control input in steady-state operation is meant. The divergence of the output from the value determined by the control input is called steady-state operation error. In the process of the calculation of the steady-state operation, error the quality of the transient process is compared with the **desired quality index** [2]**.**

## 3.1. Stability of the system

The dynamics of an automatic control system is characterised by the transient process following the violation of its equilibrate or steady state. Deviation from the equilibrate state is induced by the change if the system has input signal(s). A linear automatic control system could be stable or unstable.

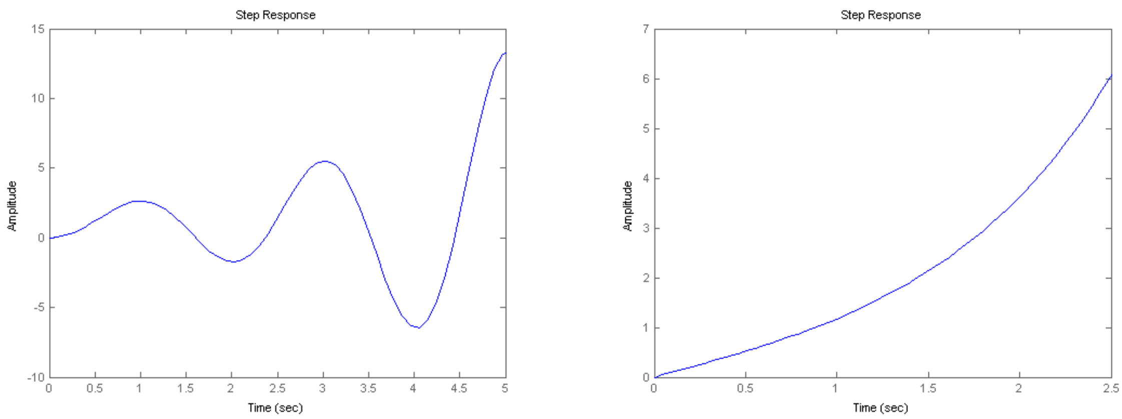
The system is **stable,** if after the transient process the steady state of the automatic control system recovers within a finite time interval. Thereby the stable system could equilibrate in a new state, if the reason of the deviation preserves, or in the previous state if the reason of the deviation elapses, or when the system is invariant in relation to the given input signal.

The system s **unstable,** if during the transient process the equilibrate state does not restore. By the deviation from the equilibrate state the oscillation of the state variables with growing amplitude might occur, or the state variables diverge monotonously from the steady state.




a) b)

Figure 3.1.Transient characteristics of a stable system a) balancing in a new state b) balancing in previous state




a) b)

Figure 3.2. Transient characteristics of an unstable system a) the variable sets to oscillate with growing amplitude b) the variable diverges monotonously from the equilibrate state

### 3.1.1. Stability criteria

#### 3.1.1.1. Determination of the stability by the transfer function

If the transfer function is given in form [2]

$$W(p)=\frac{b_0\cdot p^n+b_1\cdot p^{n-1}+...+b_{n-1}\cdot p+b_n}{a_0\cdot p^m+a_1\cdot p^{m-1}+...+a_{m-1}\cdot p+a_m}\ , \tag{3.1}$$

then the equation

$$a_0\cdot p^m+a_1\cdot p^{m-1}+...+a_{m-1}\cdot p+a_m=0 \tag{3.2}$$

is called characteristic equation of the transfer function (3.1), based on the solutions of which it is possible to decide about the stability of this system described and evaluate other quantities characterising the transient process.

The solutions of the equation (3.2) could be expressed in the form

$$p_i=\Re[p]\pm j\cdot\Im[p]\ , \tag{3.3}$$

where R – real part of the solution

I – imaginary part of the solution

And knowing, that the solution of the differential equation, describing the system

$$b_0\cdot\frac{d^n}{dt^n}\cdot y(t)+b_1\cdot\frac{d^{n-1}}{dt^{n-1}}\cdot y(t)+...+b_n\cdot y(t)=a_0\cdot\frac{d^m}{dt^m}\cdot s(t)+a_1\cdot\frac{d^{m-1}}{dt^{m-1}}\cdot s(t)+...+a_n\cdot s(t) \tag{3.4}$$

could be presented in the form

$$y(t)=y_c(t)+y_d(t)\ , \tag{3.5}$$

where $y_c(t)$ – the solution describing forced motion of the variable

$y_d$ (t) – the solution describing free motion or transient process of the variable

Placing the solutions of the characteristic equation (3.3) into the general form of the free motion

$$y_d(t)=\Sigma C_i \cdot e^{p_i \cdot t} \tag{3.6}$$

the last, in case of real solution, expresses as follows

$$y_d(t)=\Sigma C_i \cdot e^{\Re[p]_i \cdot t} \tag{3.7}$$

and, in case of complex solutions, as follows:

$$y_d(t)=\Sigma C_i \cdot e^{\Re[p]_i \cdot t} \cdot \sin(\Im[p]_i \cdot t+\varphi_i) \ . \tag{3.8}$$

It is known from the previous division, that the system is stable if the state variable equilibrates or in a new or in the previous state or, the free movement of the system grows down to zero. Mathematically presented

$$\lim_{t\to\infty} y_d(t)=0 \ , \tag{3.9}$$

on the bases of which it is, possible to conclude from the equations (3.7) and (3.8), that for each solution of the characteristic equation is valid

$$\Re[p]_i<0 \tag{3.10}$$

Alternatively, the system is stable if the real parts of all solutions of the characteristic equation are negative

Example *3.1.*

Transfer function of the $PT_1$-block is

$$W=\frac{K}{1+T\cdot p}$$

And its characteristic function

$$1+T\cdot p=0 \ .$$

The solution of the characteristic function is

$$p=-\frac{1}{T} \ ,$$

On the ground of which it could be confirmed, that the $PT_1$-block is stable by any values of its parameters.

*Example 3.2.*

The transfer function of the I-block is

$$W=\frac{K}{p}$$

And its characteristic equation

$$p=0\ .$$

On this basis it is possible to confirm, that the I-block is substantially unstable

For easier survey, the solutions are presented graphically on the complex plane, which is called p-plane. Using the facilities of the mathematical software, it is easily possible to add there the axes of damping and characteristic frequencies. The damping rate could be read by the radiuses from the zero of the p-plane, the values of which correspond to the sine of the angle between the radius and the imaginary axe. Fur reading the radian frequency one has to read the distance of the point from the zero point.

*Example 3.3.*

The transfer function of the $PT_3$-block is given in the form

$$W=\frac{100}{0.004\cdot p^3+0.06\cdot p^2+0.3\cdot p+1}\ .$$

Determining the poles of the transfer function (zeros of the characteristic equation) and locating the results on the p-plane one gets as result the pole diagram presented in the figure 3.3.

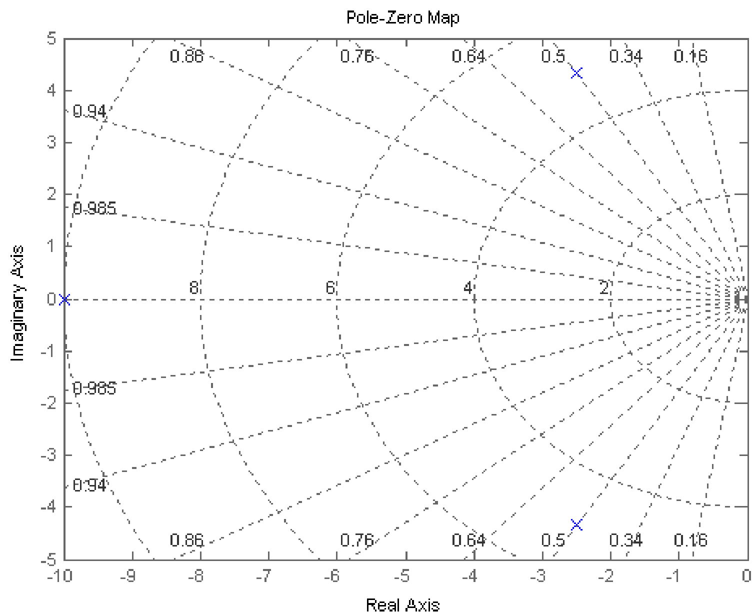


Figure 3.3. Pole-zero diagram

So as one has to do here with the characteristic equation of third order, one has to deal with three poles also, which are presented by crosses on the diagram. One of which is placed on the real axe, other describe conjugate complex values, that enables to conclude that one has to do with an oscillating process with damping ratio 0.5 and radian frequency 5 $s^{-1}$. However, system itself is stable, so as real parts of all poles are negative.

### 3.1.1.2. Routh's stability criterion

E. J. Routh has formulated the criteria of system stability, by simultaneous implementation of which it is possible to assert, that the system is stable [2], [9]:

The necessary but not sufficient condition for the system stability is that all coefficients of the characteristic equations should have the same sign (positive) or $a_i > 0$. If required, the whole characteristic equation has to be multiplied by -1;

Necessary and sufficient condition for the system stability is that all Routh's coefficients should be positive or $R_i > 0$.

The general form of the characteristic equation was given with equation (3.2)

$$a_0 \cdot p^m + a_1 \cdot p^{m-1} + \ldots + a_{m-1} \cdot p + a_m = 0$$

On the assumption of the general form of the characteristic equation, the following algorithm determines the Routh's coefficients

$$\begin{array}{lllll}
R_0=a_0 & a_2 & a_4 & \dots & 0 \\
R_1=a_1 & a_3 & a_5 & \dots & 0 \\
R_2=a_2-\frac{R_0}{R_1}\cdot a_3 & a'_4=a_4-\frac{R_0}{R_1}\cdot a_5 & a'_6=a_6-\frac{R_0}{R_1}\cdot a_7 & \dots & 0 \\
R_3=a_3-\frac{R_1}{R_2}\cdot a'_4 & a'_5=a_5-\frac{R_1}{R_2}\cdot a'_6 & a'_7=a_7-\frac{R_1}{R_2}\cdot a'_8 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_m=a_m & 0 & 0 & 0 & 0
\end{array} \tag{3.11}$$

*Example 3.4.*

Let be given differential equation describing a system.

$$0.045\frac{s^3}{m}\cdot\dddot{y}(t)+2.78\frac{s^2}{m}\cdot\ddot{y}(t)-78.6\frac{s}{m}\cdot\dot{y}(t)-897\frac{1}{m}\cdot y(t)=0.8\frac{1}{V}\cdot s(t)+0.027\frac{1}{T}\cdot n(t) \ .$$

Based on the Lap lace transformation, handled in the chapter two, the characteristic equation could be expressed as

$$0.045\frac{s^3}{m}\cdot p^3+2.78\frac{s^2}{m}\cdot p^2-78.6\frac{s}{m}\cdot p-897\frac{1}{m}=0 \ ,$$

Which will be at first studied with the Routh's first criterion

$a_0>0 \ \rightarrow \ 0.045>0$ The condition met;

$a_1>0 \ \rightarrow \ 2.78>0$ The condition is met

$a_2>0 \ \rightarrow \ -78.6<0$ The condition is not met!

On this bases on might conclude that the system is not stable

### 3.1.1.3. Hurwitz's stability criterion

A. Hurwitz simplified the Routh's criterion and formulated it as follows: [2], [9]:

> Necessary but not sufficient condition for the stability of a system is, that all coefficients of the characteristic equation should be with the same sign (positive) or $a_i > 0$. If required the whole characteristic equation must be multiplied by –1.;
>
> Necessary and sufficient condition for the system stability is that Hurwitz's determinant and its diagonal minors must be positive.

Hurwitz's determinant is composed from the coefficients of the characteristic equation as follows:

$$\Delta=\begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_m \end{bmatrix}>0 \qquad (3.12)$$

In addition, the diagonal minors correspondingly

$$\Delta_1=[a_1]>0 \ , \qquad (3.13)$$

$$\Delta_2=\begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}>0 \ , \qquad (3.14)$$

$$\Delta_3=\begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}>0 \ \text{etc..} \qquad (3.15)$$

*Example 3.5.*

Let the system given in the Example 3.ă be upgraded with a PD-regulator, after what the differential equation will have the following form (in condition that the disturbances are not considered)

$$0.045\frac{s^3}{m}\cdot\dddot{y}(t)+2.78\frac{s^2}{m}\cdot\ddot{y}(t)+61.2\frac{s}{m}\cdot\dot{y}(t)+234\frac{1}{m}\cdot y(t)=0 \ ,$$

Which could be considered now as characteristic equation and investigate it with the Hurvitz'a first criterion

$a_0>0 \ \rightarrow \ 0.045>0$ The condition met

$a_1>0 \ \rightarrow \ 2.78>0$ The condition is met

$a_2>0 \ \rightarrow \ 61.2>0$ The condition is met

$a_3>0 \ \rightarrow \ 234>0$ The condition is met

Based on this, one might study the system based on the second criterion

$\Delta=\begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}>0 \ \rightarrow \ 125.14>0$ The condition is met

$\Delta_1 = [a_1] > 0 \;\; \rightarrow \;\; 2.78 > 0$ The condition is met

Herewith it might be concluded, that the system is stable

#### 3.1.1.4. Nyquist's stability criterion

Nyquist's stability criterion belongs to the frequency methods, where the frequency characteristic curves of switched-off feedback or open loop systems are investigated, that enables the stability of systems with delay blocks to be assessed, what has not been possible by Routh nor by Hurwitz. However, the mathematical presentation of the Nyquist general criterion is complicated and therefore one special case of this criterion is more often applied, on the bases of which it is possible to evaluate the stability of predominately used automatic control systems, however, the results should be taken with some reservation. The special case is called the Nyquist's left-hand rule, that could be applied to the system with negative feedback and it is formulated as follows: [2], [3], [6], [8]:

> The necessary but not sufficient stability condition of a closed system is, that the radian frequency curve of the open system does not embrace the point (-1, j0) on the complex plane.

It means, that when moving on the radian frequency curve from the frequency 0 to ∞ direction, the Nyquist's point (-1, j0) remains to the left from the radian frequency curve. So, as here one has to do with sufficient condition only, then it is recommended to check the result with some other method.

To obtain the equation of the radian frequency curve the following transformation will be made in the transfer function:

$$p = j\omega \;, \tag{3.16}$$

As a result of what the equation of the radian frequency curve could be presented in generic form:

$$W(j\omega) = \frac{b_0 \cdot (j\omega)^n + b_1 \cdot (j\omega)^{n-1} + \ldots + b_{n-1} \cdot (j\omega) + b_n}{a_0 \cdot (j\omega)^m + a_1 \cdot (j\omega)^{m-1} + \ldots + a_{m-1} \cdot (j\omega) + a_m} \;. \tag{3.17}$$

In addition, the transfer function of the open loop cold is determined according to the generic structure of the automatic control system, presented in the figure 3.4.

$$W_0 = W_R \cdot W_P \,, \tag{3.18}$$

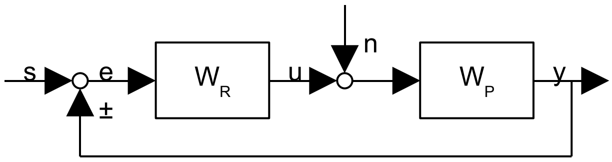



Figure 3.4. Generic structure of an automatic control system

That also could be presented in the form of equation (3.17), which in turn is presented as

$$W_0 = \Re[W_0(j\omega)] + j \cdot \Im[W_0(j\omega)] \,, \tag{3.19}$$

On the bases of what the Nyquist's diagram on the complex plane will be composed.

*Example 3.6.*

In the figure 3.5. A structure consisting of three $PT_1$-blocks is represented

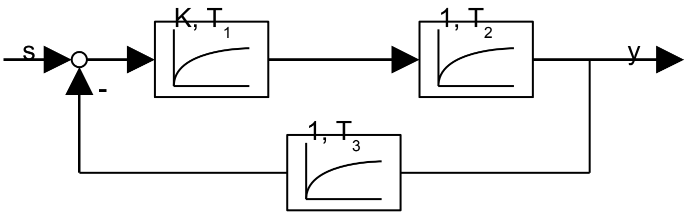
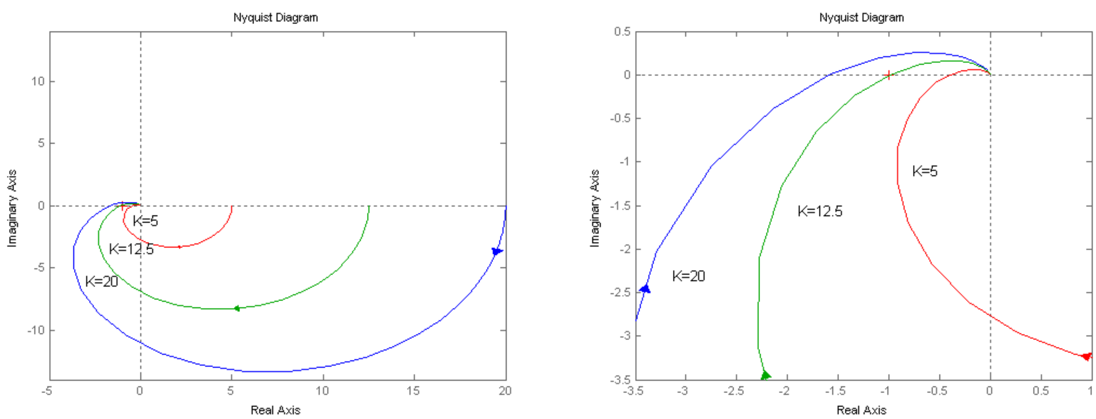



Figure 3.5.Automatic control system

Transfer function of the opened structure of witch is expressed as

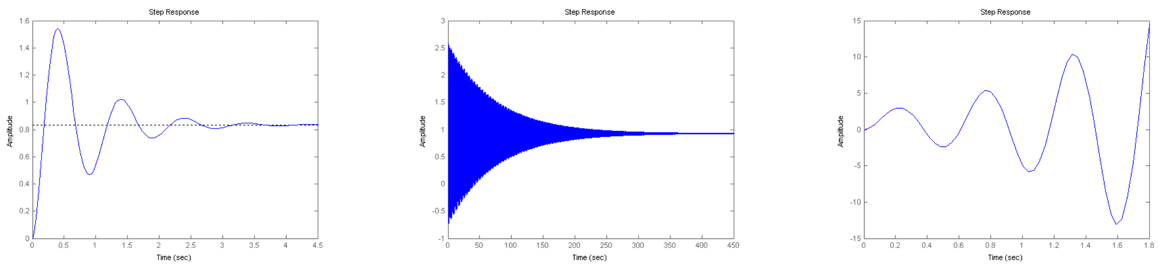
$$W_0(j\omega) = \frac{K}{(1+0.1\, s \cdot j\omega) \cdot (1+0.5\, s \cdot j\omega) \cdot (1+0.2\, s \cdot j\omega)} \,.$$




a) b)

Figure 3.6. Nyquist's diagram. a) full diagram, b) blow-up around the Nyquist's point

For the confirmation of the conclusion in the figure l 3.7 the step response of the closed system is given




a) b) c)

Figure 3.7. Step response of the closed system a) K = 5, b) K = 13,5, c) K = 20.

The results approve the stability properties stated by Nyquist, whereby in the figure 3.7.b) on has to deal with a stable system, but the quality of the transient process does not meet the requirements

### 3.1.1.5. Stability criteria on the basis of Bode diagram

This method is based on the Nyquist's stability criterion, the results of which are presented on the Bode diagram and concluded from it formulation is [2], [3], [6], [8]:

> Necessary but not sufficient stability condition of closed system is, that the ordinate $\varphi(\omega)$ or phase angel at the cutting frequency $\omega_L$ of the frequency axe on the Bode diagram of an open system is less than |180°|. by absolute value.S

As cutting frequency is the frequency considered, by which the ordinate L(ω) of the frequency axe of the logarithmic amplitude equals zero.

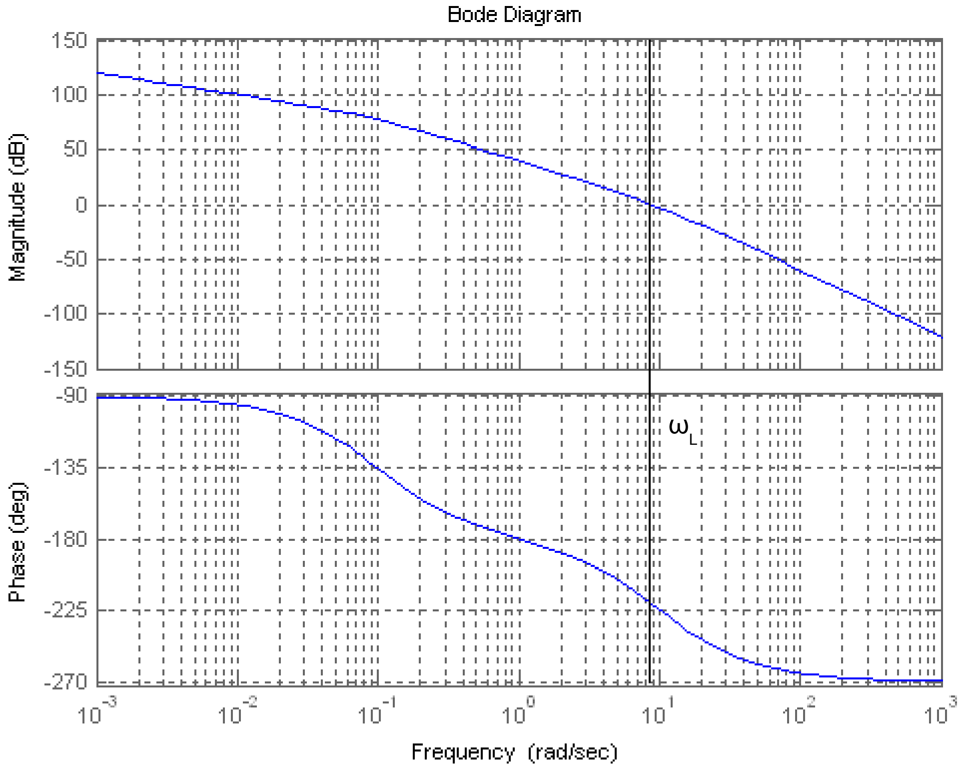



Figure 3.8. Bode diagram of an unstable system

### 3.1.1.6. Other methods

With the development of the computer technology a number of different stability determination method are remained in history, more widely known from them might have been Mihhailov's hodograph and Nichols's diagram.

Mihhailov has proceeded from the closed characteristic equation, that was solved by frequency method and the values of which were placed on the complex axes. Proceed from that Mihhailov has formulated a stability criterion of its own name [2]:

> System is stable, when by variation of the frequency ω = 0...∞ the hodograph (a curve on the plane describing the solutions of the function) crosses on the

complex plane in the positive direction in series n quadrants, where n is the order of the differential equation (characteristic equation) of the system in consider.

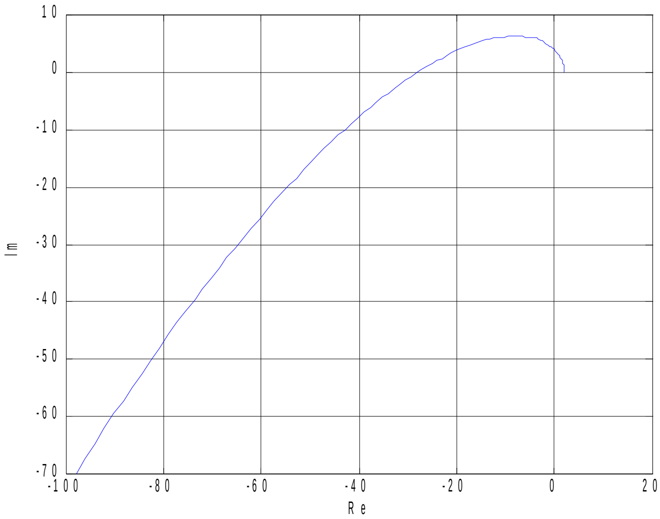



Figure 3.9. Mihhailov's hodograph of a stable 3-rd order system

Nichols's dia ted in another form of this d

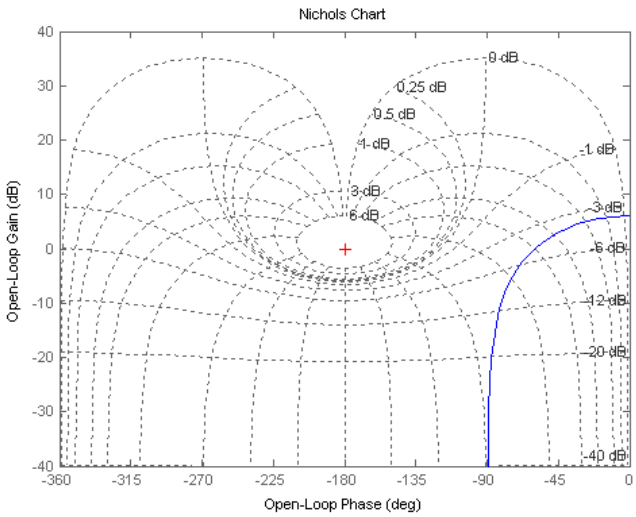



Figure 3.10. Nichols's diagram of a stable system

### 3.1.2. Structural stability

**Structural stable a**re called systems which could be convert stable by changing their parameters, whereby the structure of the system and character of blocks do not need to be changed. System, which could not be stabilised by changing its parameters (time constants or gains) are called **structurally unstable** [2]

. *Example 3.7.*

Let it be given the characteristic equation of a feedback serial connection of one $PT_1$ and two I-blocks

$$T_1 \cdot p^3 + p^2 + K_1 \cdot K_{I2} \cdot K_{I3} = 0 \ ,$$

Which when investigating by Hurwitz could be expressed as

$$a_1 \cdot 0 - a_0 \cdot a_3 < 0$$

Alternatively, given system is unstable by all possible values of its parameters, i.e. the system is structurally unstable.

*Example 3.8.*

Let it be given the characteristic equation of a feedback system of three $PT_1$ blocks

$$T_1 \cdot T_2 \cdot T_3 \cdot p^3 + (T_1 \cdot T_2 + T_2 \cdot T_3 + T_3 \cdot T_1) \cdot p^2 + (T_1 \cdot T_2 \cdot T_3) \cdot p + K_1 \cdot K_2 \cdot K_3 = 0$$

And by Hurwitz

$$a_1 \cdot a_2 - a_0 \cdot a_3 = ?$$

Or, if the system is unstable by some values of its parameters, then it is possible to stabilise it changing its parameters, not changing structure of the system, or one has to do with structurally stable system

### 3.1.3. Determination of parametric stability domains

After stability control, it is often required to analyse the impact of some parameters to the system properties. For instance, it is required to explicate how to impose certain parameters to stabilise an unstable system. In addition, it could be needed to study how much could be changed the gain of the system or of a part of the system for system remaining stable.

From the above it is known, that the coefficients of the characteristic equation of the system are determined by the parameters of the automatic control system, i.e. transfer rates and time constants. In turn, the solutions of the characteristic equation are determined by the coefficients of the equations. Thus, there exist steady dependence between solutions of the characteristic equation and system parameters.

Representing the solutions of the characteristic equation on the complex plane and if the solutions are located left to the imaginary axe, then the system is stable. Shifting of some element to the right from the imaginary axe makes system unstable. Hence, on the complex plane of the solutions the imaginary axe is the margin between stable and unstable domains.

Similarly could the margin curve of a stability characterising domain be determined in the coordinates of the characteristic equation's coefficients. While changing of all coordinates, it proceeds, in accordance with the order of the system, in the n-dimensional space. It could be determined a surface in this space, that is called D-surface. In one side from the surface is the space where the system is stable, in another side the space where the system is unstable. The D-margin is determined by the values of the coefficients of the characteristic equation, which correspond to the stability margin. If some of the characteristic equation's coefficients of a stable system should exceed the crucial value matching the stability margin, the system turns out unstable.

D-margin could be determined for the parameters that contain in the coefficients of the system. Usually the D-margin is drawn for one or two parameters.

*Example 3.9.*

Let the following equation describe the dependence between the control voltage of the magnets of a monorail s (t), the air gap between the train and the rail x (t); and the asperity of the path n (t):

$$0.445\cdot\dddot{y}(t)+1.72\cdot\ddot{y}(t)-288.3\cdot\dot{y}(t)-2241\cdot y(t)=0.5\cdot s(t)-K_n\cdot n(t)$$

Appling the Hurwitz criterion it could be concluded, that one has to do with an unstable system

$a_0=0.445>0$ Condition met

$a_1 = 1.72 > 0$ Condition met

$a_2 = -288.3 < 0$ Condition net

Therefore, as the parameters of this system could not be changed, then one has to do with structurally unstable system. In order to make this system usable a stable it will be upgraded with negative feedback and PD-regulator, after which the system turns to be structurally stable. Differential equation of the regulator would be

$s(t) = K_R \cdot (s_R(t) + T_D \cdot \dot{s}_R(t))$ ,

But, taking into account that the output f the system is regulator's input and the input of the system could be considered as regulator's output, the differential equation of the regulator, negative feedback in consider, presents itself in the form

$s(t) = -K_R \cdot (y(t) + T_D \cdot \dot{y}(t))$ .

New differential equation of the system will be

$$0.445 \cdot \dddot{y}(t) + 1.72 \cdot \ddot{y}(t) - 288.3 \cdot \dot{y}(t) - 2241 \cdot y(t) = 0.5 \cdot [-K_R \cdot (y(t) + T_D \cdot \dot{y}(t))] - K_n \cdot n(t)$$

,

Which after arrangements could be represented in the form

$$0.445 \cdot \dddot{y}(t) + 1.72 \cdot \ddot{y}(t) + (-288.3 + 0.5 \cdot K_R \cdot T_D) \cdot \dot{y}(t) + (-2241 + 0.5 \cdot K_R) \cdot y(t) = -K_n \cdot n(t)$$ ,

From the characteristic equation of which, in accordance with the Hurwitz's stability criterion, could be expressed:

$a_0 = 0.445 > 0$ Condition met

$a_1 = 1.72 > 0$ Condition met

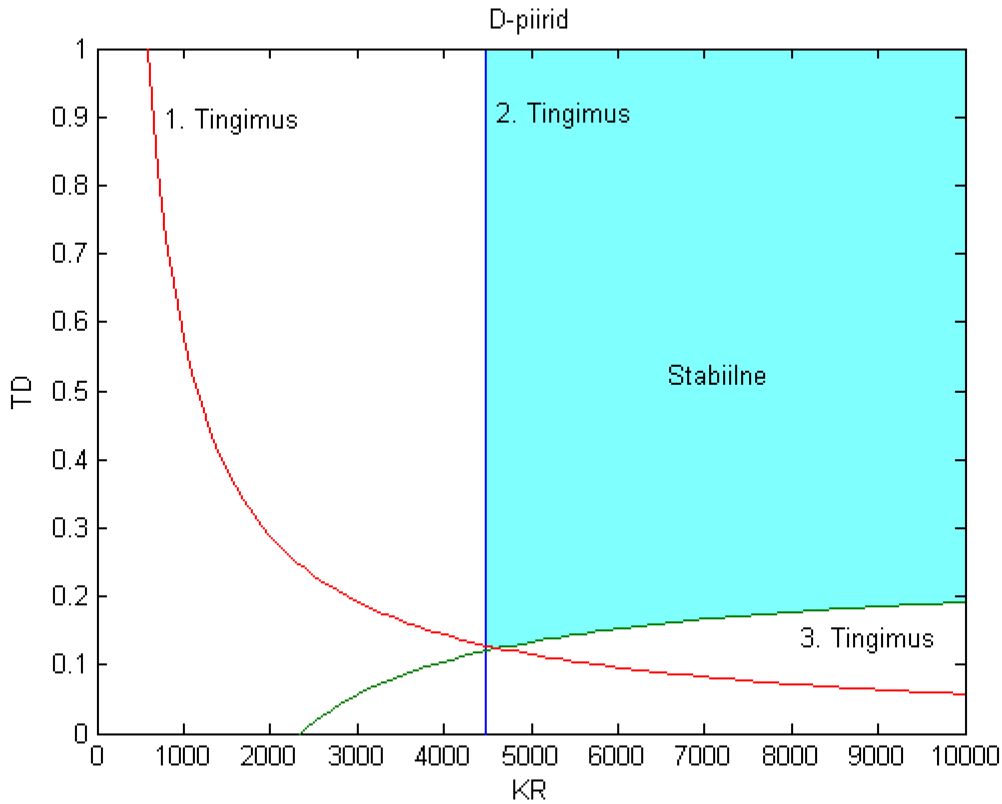
$a_2 = -288.3 + 0.5 \cdot K_R \cdot T_D > 0 \Rightarrow T_D > 576.6 \cdot \frac{1}{K_R}$ 1. condition

$a_3 = -2241 + 0.5 \cdot K_R > 0 \Rightarrow K_R > 4482$ 2. condition

$\Delta_1 = [a_1] = [1.72] > 0$ Condition met

$\Delta = \begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix} > 0 \Rightarrow a_1 \cdot a_2 - a_0 \cdot a_3 > 0 \Rightarrow T_D > 0.25 - 583 \cdot \frac{1}{K_R}$ 3. condition

Presenting three appointed conditions graphically it is easy to determine the stability domain, which is featured in the figure 3.10 as colored domain. If to select the parameters of the regulator from inside of this domain, one gets a stable system for sure.

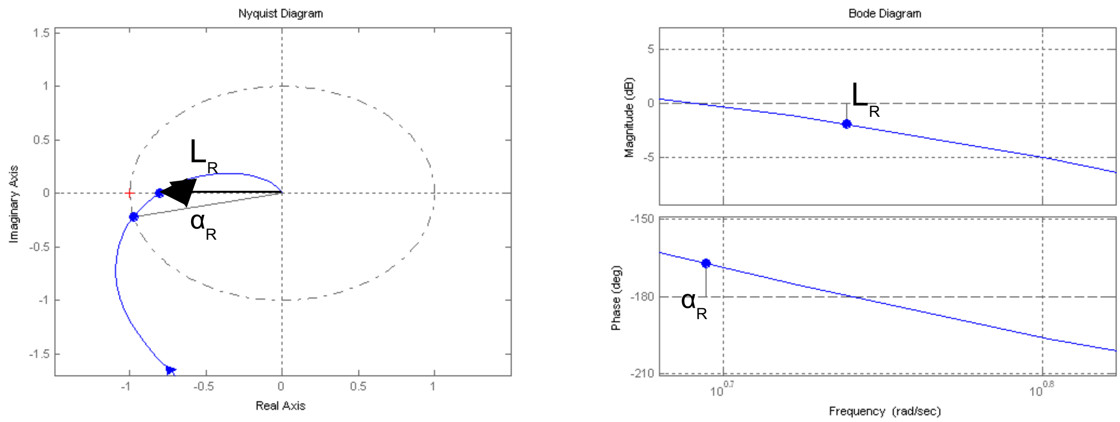



margin could be determined, exceed of which a stable system turns out unstable. Also, describing an automatic control system by equations one has to consider some errors due to the following facts

- Simplifications are made by composing of the equations
- Instead of non-linear equation the linear one are used
- System parameters are determined approximately, i.e. with certain error
- System parameters could change n time

Therefore, theoretically stable system could proved unstable while implementing in practice, especially in case if its calculated characteristic curves are too close to the stability margin. It follows from this, that the parameters of a real system should be selected so, that the system would have a certain stability reserve. Depending on the stability criterion different indexes could be used to characterise the stability reserve.

For example, by Nyquist's criterion as amplitude reserve, as well phase reserve is determined. It could be done using Bode diagram as well.




a) b)

Figure 3.11. Stability reserve for a $PT_3$-element a) by Nyquist b) by Bode (enlarged image)

The amplitude reserve and phase reserve are considered mainly so as it is possible on ttheir basis determine the system parameters. For instance, if it is wanted to increase the amplitude reserve not changing the phase reserve, then one has to change the gain f the system (remaining –p-block does not impact on the phase). The relevance of these characterising parameters the following sequence of thoughts could be explained:

The input signal of the system is sine signal with a frequency by which the gain of the system is $K \geq 1$ and phase shift is exactly 180°. If the output signal-will be directed to the input of the system by negative feedback, both signals merge, in result of which the input sign of the system increases, although the set value has not changed, and this process continues with continuing amplification, or the system will “generate” or one has to do with an unstable process.

## 3.2. Evaluation of the transient process’s quality

The system is mathematically or theoretically stable if the transien6t process is damping during a finite time interval. It does not follow from that, any stable process is suitable for practical application. Practically unsuitable are, for instance, the systems, where the restoration time of stable state is longer than permissible, or the amplitudes of the oscillating transient process are too high. In general, the automatic control structure is applicable, if its transient process is of **required quality**

Fur the characterisation of the transient process the following main indexes are used: [2]:

Static or permanent operation error ε;

Duration of the head front $t_r$;

Duration of the transient process $t_s$;

Maximal overregulation σ and corresponding to it time $t_m$;

Vibration or the number of half-waves during the oscillating transient process

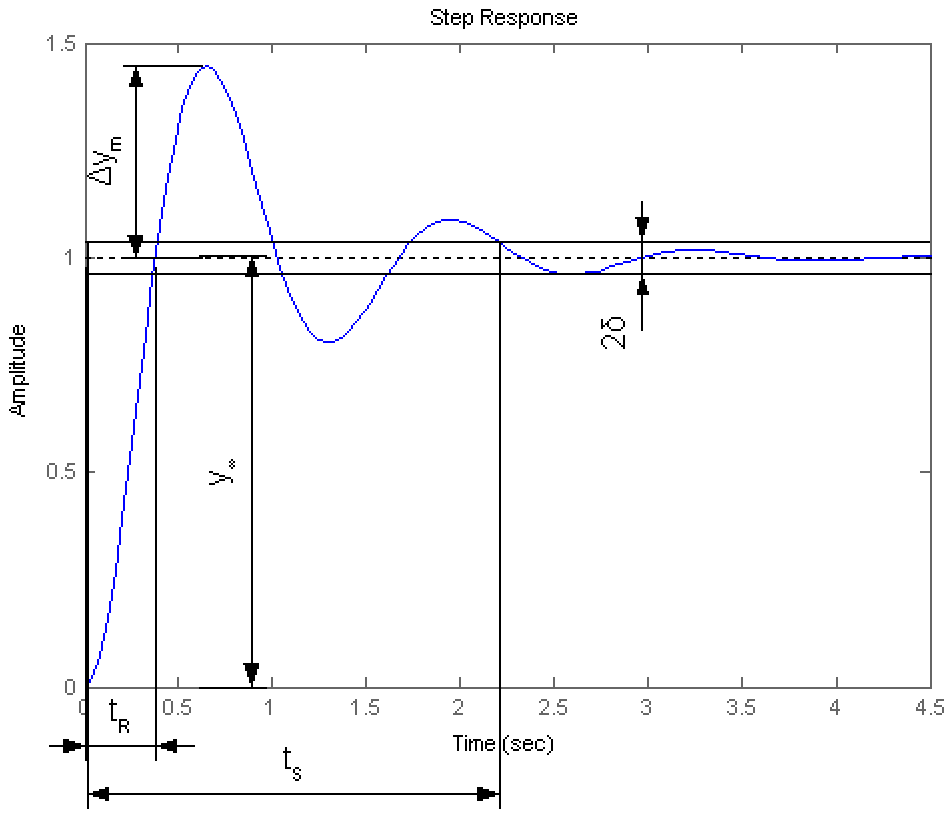
**Static** or **permanent operation error** ε characterises the accuracy of the system and is the difference of the settled state of the transient process, and of the steady state determined by the control action. Systems, static error of which is zero are called astatic systems.

**Duration of the head front** $t_r$ is considered the time interval what is needed for changing of the signal response in the interval (0 or 0,1 ... $y_\infty$-δ).

**Tolerance** δ is called the permitted deviation from the steady state determined by the control action

**Duration of the transient process** t is called the time period$_s$ after which the transient process has reduced into the tolerance limits

The duration of the heat front and total duration of the transient proceed characterise the action speed of the system. The last has great importance in those systems, in the




**Maximal over regulation** σ is determined by the relation

$$\sigma = \frac{\Delta y_m}{y_\infty} \ . \tag{3.20}$$

Maximal over regulation determines maximal admissible amplitude of the transient process, which is stated starting from the operational feature of the system. Let it be said here, that in most cases the duration of head front and maximal overregulation are related reciprocally.

The regulating quality could be assessed or experimentally or by calculated signal response, whereby in the calculations both could be used as analytic as well numerical and indirect methods.

The grey area in the figure 3.13. characterises the deviation of the output in relation to the input. In case of an ideal transient characteristic, the grey are would be absent, or the ouput would follow the input, whereby the measure of difference would be the gain only. To evaluate the regulating quality in this process the application of the integral criteria is suitable, in the run of which the differences between ideal transient characteristic and of the real one are assessed, or simpler, the area of the grey region will be evaluated. For this evaluation the following calculation methods could be used:

Integral of the absolute error or IAE-criterion

$$J = \int_0^\infty |y(t) - y_\infty| dt\ ; \tag{3.21}$$

Integral of the quadratic error or ISE-criterion

$$J = \int_0^\infty |y(t) - y_\infty|^2 dt\ ; \tag{3.22}$$

Integral of the time-weighted absolute error or ITAE-criterion

$$J = \int_0^\infty t \cdot |y(t) - y_\infty| dt\ ; \tag{3.23}$$

Integral of the time-weighted quadratic error or ITSE-criterion

$$J = \int_0^\infty t \cdot |y(t) - y_\infty|^2 dt\ . \tag{3.24}$$

For processes, real transient characteristic of which intersects the ideal transient characteristic, the quadratic methods should be used, so as the signs of the error is changing in the process, with, if using absolute methods, could lead to wrong assessments.

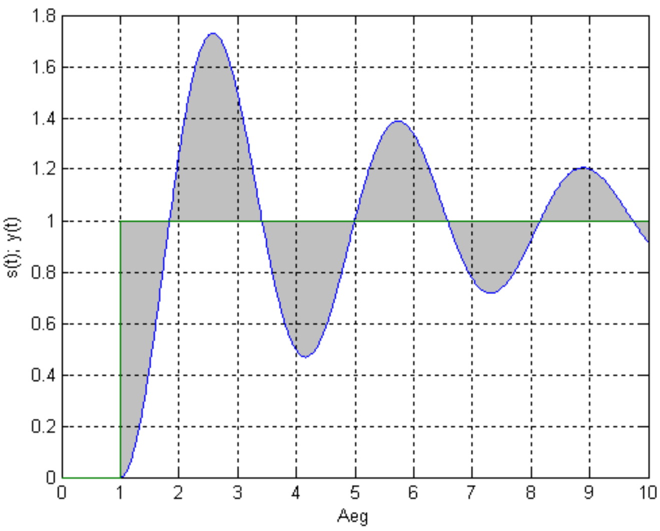


Figure 3.13.Difference of input and output of a $PT_2$-block

### 3.2.1. Determination of the calculated permanent operation error

The determination of the settled output value and permanent operation error by calculations is needed for the evaluation of the accuracy of the system under design even before the simulation, which reduces the scope of work in development. In the figure 3.14. The general form of an automatic control system is represented. For this system one can write [3]:

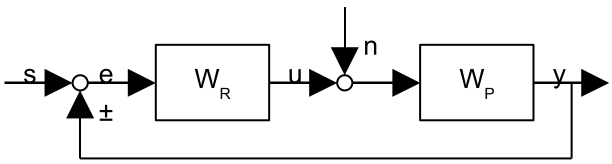



Figure 3.14. Automatic control system

$$y(p)=\frac{W_R\cdot W_P}{1\mp W_R\cdot W_P}\cdot s(p)+\frac{W_P}{1\mp W_R\cdot W_P}\cdot n(p)\ . \tag{3.25}$$

For the determination of the settled value the equation (3.25) will be solved as follows:

$$y_\infty=\lim_{p\to 0} p\cdot\left[\frac{W_R\cdot W_P}{1\mp W_R\cdot W_P}\cdot\frac{s_\infty}{p}+\frac{W_P}{1\mp W_R\cdot W_P}\cdot\frac{n_\infty}{p}\right]\ . \tag{3.26}$$

Three solutions of most widespread process-regulator combinations (3.26) are presented in the table 3.1

Table 3.1. Settled output values of the system and permanent operation error related to the control input

| ***Regulator*** | ***Process*** | ***Settled value*** | ***Permanent operation error*** |
|---|---|---|---|
| P | $PT_n$ | $y_\infty=\frac{K_R\cdot K_P}{1+K_R\cdot K_P}\cdot s_\infty+\frac{K_P}{1+K_R\cdot K_P}\cdot n_\infty$ | $\frac{1}{1+K_R\cdot K_P}$ |
| | $IT_n$ | $y_\infty=s_\infty+\frac{1}{K_R}\cdot n_\infty$ | 0 |
| I | $PT_n$ | $y_\infty=s_\infty+0\cdot n_\infty$ | 0 |
| | $IT_n$ | Structurally unstable | |

Permanent operation error is given in the operator form as follows;

$$\varepsilon(p)=s(p)-y(p)\ , \tag{3.27}$$

which, considering the equation (3.25) takes the form

$$\varepsilon(p)=\frac{1}{1+W_R\cdot W_P}\cdot s(p)-\frac{W_P}{1+W_R\cdot W_P}\cdot n(p)\ . \tag{3.28}$$

Solving the equation (3.28) by the condition

$$\varepsilon=\lim_{p\to 0} p\cdot\left[\frac{1}{1+W_R\cdot W_P}\cdot\frac{s_\infty}{p}-\frac{W_P}{1+W_R\cdot W_P}\cdot\frac{n_\infty}{p}\right]\ . \tag{3.29}$$

On the basis of these calculations one might conclude, that in the systems containing one I-block it is possible to reduce the permanent error to zero, Which is not possible with P-regulator

## Exercises to chapter 3.

*Exercise 5.*

Inspect the stability of the if the transfer function is presented as follows:

$$W(p)=\frac{(p+2)\cdot(p-4.2)\cdot(p+7.1)}{(p+2.5)\cdot(p+1)^2\cdot(p+2)\cdot(p-1)}\ .$$

*Exercise 6.*

Verify the stability of the system by Routh’s criterion if the characteristic equation of the system is the following

$$0.0076\cdot p^5+0.051\cdot p^4+0.17\cdot p^3+9.67\cdot p^2+78.1\cdot p+542=0\ .$$

*Exercise 7.*

Characteristic equation of a system controlled with two P-regulators is:

$$T^2\cdot p^3+2\cdot T\cdot p^2+(K\cdot K_R+1)\cdot p+K\cdot K_I\cdot K_{R1}\cdot K_{R2}=0\ .$$

Determine the stability domain by the regulators $K_{R1}$ and $K_{R2}$, if the parameters of the system are K = 1, $K_I$ = 0.5 $s^{-1}$ and T = 2 s.

*Exercise 8.*

Determine the stability domain of the regulator, if the characteristic equation is given in the form

$$[T_I \cdot T_P \cdot (1 - K_P \cdot K_R)] \cdot p^2 + [T_I + K_P \cdot K_R \cdot (T_I - T_S)] \cdot p + K_P \cdot K_R = 0 \quad .$$

*Exercise 9.*

Appoint the following parameters from the near by graphic

Joonis 3.15. Siirdekarakteristik

- Permanent operation error;
- Maximal over regulation;
- Approximate IAE-criterion;
- Approximate ISE-criterion.